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Chiral field theories, Konishi anomalies and matrix models

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Abstract: We study a chiral $\mathcal{N}=1$, U(N) field theory in the context of the Dijkgraaf-Vafa correspondence. Our model contains one adjoint, one conjugate symmetric and one antisymmetric chiral multiplet, as well as eight fundamentals. We compute the generalized Konishi anomalies and compare the chiral ring relations they induce with the loop equations of the (intrinsically holomorphic) matrix model defined by the tree-level superpotential of the field theory. Surprisingly, we find that the matrix model is well-defined only if the number of flavors equals two! Despite this mismatch, we show that the $1/\hat{N}$ expansion of the loop equations agrees with the generalized Konishi constraints. This indicates that the matrix model — gauge theory correspondence should generally be modified when applied to theories with net chirality. We also show that this chiral theory produces the same gaugino superpotential as a nonchiral SO(N) model with a single symmetric multiplet and a polynomial superpotential.

Keywords: Superstring Vacua, Anomalies in Field and String Theories, Matrix Models.

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Contents

1. Introduction

A surprising feature of $\mathcal{N}=1$ strong coupling dynamics was uncovered in the seminal work of Dijkgraaf and Vafa [1, 2, 3], who found a relation between the gaugino superpotential of a confining $\mathcal{N}=1$ theory and certain holomorphic [4] matrix models. The recipe they proposed takes the tree-level superpotential of such a theory to be the action of the dual matrix model. This conjecture was proved for a few nontrivial examples, via two distinct methods. One approach [5] uses covariant superfield techniques in perturbation theory to integrate out massive matter fields in a gaugino background. A different method was proposed in [6, 7], where it was shown that the loop equations of the matrix model coincide formally with chiral ring relations induced by certain generalizations of the Konishi anomaly.

Up to now the matrix model — field theory correspondence has been applied almost exclusively to the non-chiral case[1] [1]–[27]. The motivation of the present paper is to test the conjecture for the case of chiral models. Ideally, one would like to know if matrix models can be used to calculate effective superpotentials of SUSY-GUTs or other supersymmetric extensions of the standard model. Of course we are far from answering this question. Instead, we will study a model with gauge group $\mathrm{U}(N)$ and chiral matter content chosen to allow for a straightforward large N limit. The matter consists of a field Φ in the adjoint representation, two fields A, S in the antisymmetric and conjugate symmetric two-tensor representations, and eight fields $Q_1 \dots Q_8$ in the fundamental representation to cancel the chiral anomaly. The tree level superpotential has the form:

$$W_{\mathrm{tree}} = \mathrm{tr}[W(\Phi) + S\Phi A] + \sum_{f=1}^{8} Q_f^T S Q_f \,, \tag{1.1}$$

where W is a complex polynomial.

This model has the advantage that the number of matter fields is independent of the rank of the gauge group, thus allowing for a large N limit with fixed matter content. Further motivation to study this model is provided by its interesting type-IIA/M-theory realization [30]–[34].[2] By taking the strong coupling limit (which amounts to lifting the brane configuration to M-theory), it was argued in these references that the model is described by a smooth curve. This can be interpreted as the existence of a mass gap and confinement, a conclusion which is of course also suggested by our model's one loop beta function. The geometric engineering of such models is discussed in [36] by using methods of [25, 37, 43].

In the present paper, we study the gaugino superpotential obtained after confinement. We shall show that the effective superpotential agrees with that of a different $\mathcal{N} = 1$ theory, namely a non-chiral $\mathrm{SO}(N)$ model with a single chiral superfield X in the symmetric representation and a tree-level superpotential given by $\mathrm{tr}\, W(X)$ (the gaugino superpotential for such models was recently investigated in [18]). This relation can be understood most easily by turning on a D-term deformation of the original model, under which the theory flows at low energies to the $\mathrm{SO}(N)$ model with symmetric matter. Because the effective superpotential is protected by holomorphy, its form must be independent of the choice of Fayet-Iliopoulos term, which explains why one obtains agreement between the two theories. One can make this argument more precise by computing the first (a.k.a. Veneziano-Yankielowicz) approximation to the gaugino superpotentials upon using scale

[1]See however [28, 29] for calculations of the effective superpotential using Konishi anomaly relations in certain chiral field theories.

[2]We note that there also exist IIB brane configurations describing chiral models, for example the so-called "brane boxes" of [35], whose stability is unfortunately unclear. Geometric duals for these configurations are not currently known.

matching techniques, and we shall do so below, finding agreement. To give a complete proof of low-energy equivalence, we use the more powerful technique of chiral ring relations [7], which allows us to characterize the exact effective superpotential in terms of solutions to certain algebro-differential equations induced by generalized Konishi anomaly constraints. Then the connection between the chiral and SO(N) models follows upon matching the relevant relations in the chiral rings.

In principle, the generalized Konishi constraints derived below suffice to completely determine the exact gaugino superpotential, which can be extracted with arbitrary precision by solving the relevant equations. However, it is interesting to follow the beautiful insight of Dijkgraaf and Vafa in order to construct a holomorphic matrix model whose free energy specifies the superpotential. This can be achieved by building a matrix integral whose loop equations reproduce the Konishi constraints. Applying these ideas, one finds some novel phenomena, which are related to the chiral character of our matter representation.

In fact, the holomorphic matrix model with action given by the tree level superpotential of our chiral field theory turns out to be ill defined. The problem is that, although the matrix model action is invariant under the complexified $GL(\hat{N}, \mathbb{C})$ gauge group, the measure fails to be invariant unless the number of matrix model flavors $\hat{Q}_f$ equals two. This phenomenon, which is due to the presence of a chiral matter content, forces us to work with a matrix model which contains only two flavors, even though the associated field theory contains eight ! Then the matrix partition function is well-defined, and we show that the loop equations agree with the generalized Konishi constraints despite the mismatch in the number of flavors. Our matrix model is intrinsically holomorphic, in the sense that it does not admit a real or hermitean version. This is due to the fact that our matter representation is chiral.

Having extracted the relevant matrix model, we compare it with the model which governs the gaugino superpotential of the SO(N) theory with symmetric matter, a comparison which sheds different light on the relation between the two theories.

The paper is organized as follows. In section 2, we analyze the classical moduli space of our theory by solving the F- and D-flatness constraints. For a diagonal vev of Φ, we find that the gauge group is broken to a product $\prod_{i=1}^{d} \mathrm{U}(N_i)$, where each factor contains the same massless matter as the original theory. Computation of the leading contribution to the effective superpotential requires threshold matching, which cannot be performed directly since chirality forbids the addition of a mass term. This problem was also encountered in [28], where it was solved by deforming the superpotential in such a way as to Higgs the gauge group. We will use a similar technique. Instead of deforming the tree-level superpotential, we use independence of the effective superpotential of D-term deformations, which allows us to add a Fayet-Iliopoulos term. In the presence of a Fayet-Iliopoulos parameter (which we take to be positive), we find that the symmetric field acquires a vev, which breaks the gauge group down to SO(N) with massless matter X transforming in the symmetric representation and a superpotential $\mathrm{tr}\, W(X)$. Then the first approximation to the gaugino superpotential can be computed by standard threshold matching. The resulting nonchiral SO(N) model was recently studied in [18] via the Konishi anomaly approach of [7].

In section 3, we derive the generalized Konishi constraints for our chiral theory and show how they relate to those extracted in [18] for the SO(N) model. Since our chiral model has a different matter content, we find a different set of resolvent-like objects which enter the relevant 'loop equations'. However, we show that all such quantities are uniquely determined by the solution of a pair of equations which coincide with those derived in [18] for the SO(N) theory with symmetric matter.

Section 4 discusses the matrix model dual to our chiral theory. We show that the measure is not invariant under the central $\mathbb{C}^*$ of the $GL(\hat{N},\mathbb{C})$ gauge group (and thus the matrix model partition function vanishes or is infinite) unless the number $\hat{N}_F$ of matrix model flavors equals two. We then extract the loop equations of this model by using both the standard method of the eigenvalue representation and the approach of [22]. Finally, we discuss the relation with the matrix integral relevant for the SO(N) theory with symmetric matter.

The identifications mapping our loop equations into the Konishi constraints are given in section 5. Using this map, we extract an explicit formula expressing the gaugino superpotential in terms of the matrix model free energy. This completes the proof of the Dijkgraaf-Vafa conjecture for our case.

Section 6 presents our conclusions. In appendix A we prove gauge-invariance of the matrix model measure. Appendix B recalls the classical vacua of the SO(N) theory with symmetric matter, while appendix C contains some details relevant for the discussion of section 3.

2. A first view of field theory properties

In this section we take a first look at our field theory model. After describing it precisely, we discuss the part of the classical moduli space which will be relevant for our purpose, and give our derivation of the Veneziano-Yankielowicz superpotential, which is the first approximation to the exact glueball superpotential predicted by the Dijkgraaf-Vafa correspondence.

2.1 Description of the model

We start with a U(N) gauge group, together with chiral matter Φ, S, A in the adjoint, antisymmetric and conjugate symmetric representations, as well as N_F quarks Q_f in the fundamental representation. We consider the tree-level superpotential:

$$W_{\rm tree} = {\rm tr}\,[W(\Phi) + S\Phi A] + \sum_{f=1}^{N_F} Q_f^T S Q_f\,, \qquad (2.1)$$

where:

$$W(z) = \sum_{j=1}^{d+1} \frac{t_j}{j} z^j \qquad (2.2)$$

is a complex polynomial of degree $d+1$. We have $S^T = S, A^T = -A$ (while Φ is unconstrained) and the gauge transformations are:

$$\Phi \to U\hat{\Phi}U^\dagger\,, \qquad S \to \bar{U}SU^\dagger\,, \qquad A \to UAU^T\,, \qquad Q_f \to UQ_f\,, \tag{2.3}$$

where U is valued in $\mathrm{U}(N)$. The $\mathrm{U}(N)$ gauge symmetry is obviously preserved by $W_{\rm tree}$. Note that the fields S, A are *complex*.

We do not have quarks in the anti-fundamental representation and therefore this system is quite different from models studied in [8, 14, 15, 16]. In fact, the matter representation is chiral, in contrast to most situations previously studied in the context of the Dijkgraaf-Vafa conjecture. In particular, the model will have a chiral anomaly unless we take $N_F = 8$. In the following, we shall focus on the non-anomalous case though we allow N_F to take an arbitrary value in most formulas (this permits us to recover the anomaly cancellation constraint $N_F = 8$ as a consistency condition required by gaugino condensation).

This model can be obtained through an orientifolded Hanany-Witten construction [30, 31, 32]. It can also be realized through geometric engineering, as we discuss in a companion paper [36].

2.2 The classical moduli space

Let us study the classical moduli space of our theories. Part of the discussion below is reminiscent of that given in [37, 42] and [21] for quiver gauge theories, though of course we have a rather different matter content and hence the details are not the same.

The F-flatness constraints are:

$$\Phi^T S = S\Phi\,, \tag{2.4}$$

$$\Phi A - A\Phi^T + 2\sum_{f=1}^{N_F} Q_f Q_f^T = 0\,, \tag{2.5}$$

$$W'(\Phi) + AS = 0\,, \tag{2.6}$$

$$SQ_f = 0\,, \tag{2.7}$$

while the D-flatness condition is:

$$\frac{1}{2}[\Phi^\dagger, \Phi] + S^\dagger S - AA^\dagger - \frac{1}{2}\sum_{f=1}^{N_F} Q_f Q_f^\dagger = 0\,, \tag{2.8}$$

where the left hand side is the moment map for our representation of $\mathrm{U}(N)$.

To understand the solutions, notice that (2.6) implies:

$$W'(\Phi)^2 = (AS)^2 = ASAS\,. \tag{2.9}$$

Using the transpose $SA = W'(\Phi^T)$ of (2.6) in the right hand side gives:

$$W'(\Phi)^2 = AW'(\Phi^T)S = ASW'(\Phi) \tag{2.10}$$

where in the last equality we used equation (2.4). Applying (2.6) once again in the right hand side of (2.10), we find:

$$W'(\Phi)^2 = 0\,. \tag{2.11}$$

Let us assume that $[\Phi^\dagger, \Phi] = 0$, i.e. Φ is a normal matrix. Then Φ is diagonalizable via a unitary gauge transformation, and equation (2.11) shows that Φ can be brought to the form:

$$\Phi = \mathrm{diag}(\lambda_1 \mathbf{1}_{N_1}, \ldots, \lambda_d \mathbf{1}_{N_d}) \tag{2.12}$$

where $\lambda_1, \ldots, \lambda_d$ are the distinct roots of $W'(z)$ and $N_1, \ldots, N_d$ and non-negative integers such that $N_1 + \cdots + N_d = N$. If $N_j = 0$ for some root λ_j, we use the convention that the corresponding block $\lambda_j \mathbf{1}_{N_j}$ does not appear in (2.12).

With this form of Φ, equation (2.4) shows that S must be block-diagonal:

$$S = \mathrm{diag}(S_1, \ldots, S_d) \tag{2.13}$$

where S_j are symmetric $N_j \times N_j$ matrices. When bringing Φ to the form (2.12), we are left with a residual $\prod_{j=1}^{d} \mathrm{U}(N_j)$ gauge symmetry corresponding to the transformations $U = \mathrm{diag}(U_1 \ldots U_d)$ with $U_j \in \mathrm{U}(N_j)$. Using this symmetry, we can bring S_j to the form:[3]

$$S_j = \mathrm{diag}(0_{N_j^{(0)}}, \sigma_j^{(1)} \mathbf{1}_{N_j^{(1)}}, \ldots, \sigma_j^{(m_j)} \mathbf{1}_{N_j^{(m_j)}}) \tag{2.14}$$

where:

$$0 = \sigma_j^{(0)} < \sigma_j^{(1)} < \cdots < \sigma_j^{(m_j)}\,. \tag{2.15}$$

The multiplicities $N_j^{(k)}$ satisfy $\sum_{k=0}^{m_j} N_j^{(k)} = N_j$. In the case $m_j = 0$, we have $N_j^{(0)} = N_j$ and equation (2.14) reduces to $S_j = 0_{N_j}$. After bringing S to the form (2.14), we are left with the gauge symmetry $\prod_{j=1}^{d} \left[\mathrm{U}(N_j^{(0)}) \prod_{k=1}^{m_j} \mathrm{SO}(N_j^{(k)})\right]$.

Writing:

$$Q_f = \begin{bmatrix} Q_f^{(1)} \\ \vdots \\ Q_f^{(d)} \end{bmatrix} \tag{2.16}$$

where $Q_f^{(j)}$ is an N_j-vector, equation (2.7) requires that each $Q_f^{(j)}$ lies in the kernel of S_j. Thus we must have:

$$Q_f^{(j)} = \begin{bmatrix} q_f^{(j)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.17}$$

[3]This is the Takagi factorization (see, for example, [44] page 204) for the complex symmetric matrix S_j.

when decomposing into sub-vectors according to $N_j = N_j^{(0)} + N_j^{(1)} + \dots + N_j^{(m_j)}$. Here $q_f^{(j)}$ is a column $N_j^{(0)}$-vector. Using the remaining gauge symmetry, we bring the $N_j^{(0)} \times N_F$ matrix $q^{(j)} := \left[q_1^{(j)} \dots q_{N_F}^{(j)} \right]$ to the form:

$$q^{(j)} := \begin{bmatrix} a_1 & * & \dots & * & * & \dots & * \\ 0 & a_2 & \dots & * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{s_j} & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} , \tag{2.18}$$

where $0 < a_1 < \dots < a_{s_j}$, the symbol $*$ stands for generally distinct complex entries and the zero rows at the bottom may be absent. The rank s_j of this matrix equals the dimension of the vector space spanned by $q_1^{(j)} \dots q_{N_F}^{(j)}$, and can take values between 0 and $\min(N_j^{(0)}, N_F)$. If we let $p_j := N_j^{(0)} - s_j$, then the vevs (2.18) break the $\mathrm{U}(N_j^{(0)})$ components of the gauge group down to $\mathrm{U}(p_j)$.

Finally, equation (2.5) shows that in such a vacuum the $N_i \times N_j$ blocks of A are given by:

$$A_{ij} = \frac{2}{\lambda_j - \lambda_i} \sum_{f=1}^{N_F} Q_f^{(i)} (Q_f^{(j)})^T \, . \tag{2.19}$$

Since $Q_f^{(i)}$ lie in the kernel of S_i, this automatically satisfies condition (2.6), which in our vacuum takes the form $AS = 0 \Longleftrightarrow SA = 0$. Equation (2.19) shows that all non-vanishing entries of A lie in the $s_i \times s_j$ sub-blocks of the $N_i^{(0)} \times N_j^{(0)}$ blocks, where they are given by equations (2.19) and (2.18).

Since we assume $[\Phi^\dagger, \Phi] = 0$, the D-flatness condition (2.8) reduces to:

$$S^\dagger S - A A^\dagger - \frac{1}{2} \sum_{f=1}^{N_F} Q_f Q_f^\dagger = 0 \, . \tag{2.20}$$

Using the form of Φ, S and A discussed above (which is required by F-flatness), it is not hard to see that (2.8) implies (note that we do *not* sum over i):

$$S_i^\dagger S_i = 0 \qquad \text{for all } i \, . \tag{2.21}$$

This follows by decomposing (2.20) into $N_i \times N_j$ blocks and restricting to the case $i = j$ while using the fact that $(AA^\dagger)_{ij} = -(A\bar{A})_{ij} = -A_{ik}\bar{A}_{kj}$ and $(Q_f Q_f^\dagger)_{ij} = Q_f^i \bar{Q}_f^j$ both vanish for $i = j$ due to equations (2.16) and (2.19) except in the $N_i^{(0)} \times N_i^{(0)}$ block. This implies that the entries of the block diagonal matrix $S_i S_i^\dagger$ vanish except in this block. The entries also vanish in the $N_i^{(0)} \times N_i^{(0)}$ block because of the form (2.14).

Since $S_i^\dagger S_i$ is positive semidefinite, equations (2.21) imply $S_i = 0$ for all i and thus $S = 0$. In this case, the D-flatness condition (2.20) becomes:

$$AA^\dagger + \frac{1}{2}\sum_{f=1}^{N_F} Q_f Q_f^\dagger = 0\,, \tag{2.22}$$

which implies $A = 0$ and $Q_f = 0$ for all f by semi-positivity of the left hand side.

It follows that the only classical vacua for which $[\Phi^\dagger, \Phi] = 0$ are given by Φ of the form (2.12) and $S = A = 0$ as well as $Q_f = 0$ for all f. In such a vacuum, the gauge group is broken down to the product $\prod_{j=1}^d \mathrm{U}(N_j)$.

2.3 The Veneziano-Yankielowicz superpotential

We next discuss the leading approximation to the gaugino superpotential. As we shall see below, the effective superpotential coincides with that of an $\mathrm{SO}(N)$ field theory with a single chiral superfield X transforming in the symmetric two-tensor representation, and a tree-level superpotential $\mathrm{tr}\, W(X)$.

To compute the gaugino superpotential, we need the scale(s) of the low energy theory, which are usually obtained via threshold matching. Standard threshold matching is difficult to apply to chiral theories, since a chiral tree-level action cannot contain mass terms for the chiral fields, and thus one cannot directly integrate out such fields at one-loop. This problem was encountered for a chiral model considered in [28], where it was overcome by applying threshold matching to a certain Higgs branch. Some aspects of the same issue were recently discussed in [19].

It turns out that threshold matching can be carried out in our case provided that one first deforms the theory through the addition of a Fayet-Iliopoulos term ξ (this is useful for our purpose since holomorphy dictates that the low energy scale is insensitive to D-term deformations). Such a deformation was previously considered in [30, 31, 32] and has the following effect. Concentrating on vacua with $[\langle\Phi\rangle, \langle\Phi\rangle^\dagger] = 0$, it is not hard to see that the inhomogeneous form of (2.8) (obtained by introducing $\xi \mathbf{1}_N$ in the right hand side) together with the F-flatness constraints again imply $\langle A\rangle = 0$ and $\langle Q_f\rangle = 0$ for all f. On the other hand, the symmetric field S gets an expectation value equal to the square root of ξ, which breaks $\mathrm{U}(N)$ to $\mathrm{SO}(N)$. The vev of S gives equal masses to the quarks Q_f due to the terms $Q_f^T S Q_f$. The term $\mathrm{tr}\, S\Phi A$ gives equal masses to the fields A and Y where $Y = \frac{1}{2}(\Phi - \Phi^T)$ is the antisymmetric part of Φ. All of these masses depend on the FI-term. The symmetric part $X := \frac{1}{2}(\Phi + \Phi^T)$ of the adjoint field remains massless. The fluctuations of S around its vev are 'eaten up' while giving masses to the appropriate W-bosons through the Higgs mechanism.

Eventually, one is left with an $\mathrm{SO}(N)$ theory with a symmetric tensor X and a superpotential $\mathrm{tr}\, W(X)$. By holomorphy, the scale of this $\mathrm{SO}(N)$ theory must be independent of the original FI-parameter. It is easy to check this explicitly. Let the FI-parameter be $\xi = \nu^2$ for some real ν. Then the inhomogeneous form of (2.8) shows that the vev of S equals $\pm\nu$, and we can take the plus sign without loss of generality. The massive fields are eight quark flavors, two antisymmetric tensors (A and the antisymmetric part of Φ) and the W-bosons in the symmetric representation of the low energy gauge group SO(N). This gives the scale matching relation:

$$\Lambda_0^{N-4} = \Lambda^{N-4}\nu^4\nu^{N-2}\nu^{-(N+2)} = \Lambda^{N-4} \Longrightarrow \Lambda_0 = \Lambda\,. \tag{2.23}$$

Here Λ is the scale of the chiral high energy theory (whose one-loop beta function coefficient is $N-4$) and Λ_0 is the scale of the low energy SO(N) theory. Notice that the exponent of Λ_0 is unusual in the sense that we would expect $2N-8$ for the SO(N) theory with symmetric tensor. This can be traced back to the fact that the generators in the SO(N) theory are unusually normalized. It can be seen by noticing that the index for the fundamental representation of U(N) is $\frac{1}{2}$ whereas the index for the fundamental representation of SO(N) with conventional normalization is 1 and that a fundamental of the high energy U(N) theory descends directly to a fundamental of the low energy SO(N) theory. This normalization has already been taken into account in (2.23).

We still have the deformation $\operatorname{tr} W(X)$, which leads to diagonal vevs of X of the form (2.12). The relevant vacua of the SO(N) theory with a symmetric field are discussed in appendix B. One finds that the vev of X further breaks the Lie algebra of the low energy gauge group according to [45]:

$$so(N) \to \oplus_{i=1}^{d} so(N_i)\,. \tag{2.24}$$

It is now easy to extract the scales of the different $so(N_i)$ factors (here we use conventional normalization, since we compare only SO theories):

$$\Lambda_i^{3(N_i-2)} = \left[m_{X_i}^{N_i+2} \prod_{j\neq i} m_{W_{ij}}^{-2N_j} \right] \Lambda^{2(N-4)} \tag{2.25}$$

where $m_{X_i} = W''(\lambda_i)$ and the masses of the SO(N)/SO(N_i) W-bosons are $m_{W_{ij}} = \lambda_i - \lambda_j$. Because of (2.23), we can use the high energy scale Λ in (2.25).

The Veneziano-Yankielowicz contribution to effective superpotential has the form:

$$W_{\text{eff}} = \sum_{i=1}^{d} \frac{\mathcal{S}_i}{2} \left[\log\left(\frac{\Lambda_i^{3(N_i-2)}}{\mathcal{S}_i^{N_i-2}} \right) + N_i - 2 \right], \tag{2.26}$$

where we have inserted the factor 1/2 coming from the relative normalization of the generators of U(N) and SO(N).

Observation: One can also consider turning on a negative FI-term $\xi = -\nu^2$. Taking N to be even for simplicity and assuming $[\langle\Phi\rangle^\dagger, \langle\Phi\rangle] = 0$ and $\langle Q_f\rangle = 0$ for all f, the D-flatness condition shows that the antisymmetric field A acquires a vev. Up to a gauge transformation, we can take $\langle A\rangle = \nu J$ where J is the antisymmetric invariant tensor of Sp($N/2$):

$$J = \begin{pmatrix} 0 & \mathbf{1}_{N/2} \\ -\mathbf{1}_{N/2} & 0 \end{pmatrix}. \tag{2.27}$$

This breaks the $\mathrm{U}(N)$ gauge group down to $\mathrm{Sp}(N/2)$, with the fluctuations of A 'eaten' by the W-bosons of the coset $\mathrm{U}(N)/\mathrm{Sp}(N/2)$. Let us decompose $\Phi = \Phi_+ + \Phi_-$, where $(\Phi_\pm J)^T = \pm(\Phi_\pm J)$. Then the superpotential becomes:

$$\mathrm{tr}\left[W(\Phi_+ + \Phi_-) + \nu S \Phi_+ J + Q_f \otimes Q_f^T S\right] . \tag{2.28}$$

Integrating out Φ_+ by the equation of motion of S (which gives $\Phi_+ = -\frac{1}{\nu} Q_f \otimes Q_f^T J$) results in the superpotential:

$$W_{\mathrm{low}} = W\left(\Phi_- - \frac{1}{\nu} Q_f \otimes Q_f^T J\right) . \tag{2.29}$$

Hence at low energies we have an $\mathrm{Sp}(N/2)$ gauge theory with an antisymmetric tensor and eight fundamentals interacting through the superpotential (2.29). Because of the complicated structure of W_{low}, we did not find this branch to be useful for our purpose.

3. Low energy analysis via generalized Konishi anomalies

In this section, we extract the relevant chiral ring relations of our model and compare with those of the $\mathrm{SO}(N)$ theory with symmetric matter. We shall use the method of generalized Konishi anomalies originally developed in [6, 7]. The structure of the tree level superpotential implies:

$$j \frac{\partial W_{\mathrm{eff}}}{\partial t_j} = \langle \mathrm{tr}(\Phi^j) \rangle \quad . \tag{3.1}$$

Our strategy is to extract a set of Konishi anomaly relations which allow one to solve for the generating function $T(z) = \langle \mathrm{tr}(1/z - \Phi) \rangle$ of the chiral correlators $\langle \mathrm{tr}(\Phi^j) \rangle$ appearing in the right hand side. The integration of (3.1) allows one to compute the effective superpotential up to a piece which is independent of the coupling constants t_j.

3.1 Konishi constraints for the chiral model

As discussed in [7], the loop equations of the (adjoint) one-matrix model are formally equivalent to certain chiral ring relations induced by a generalized form of the Konishi anomaly of the $\mathrm{U}(N)$ field theory with one adjoint multiplet. The argument extends to other matter representations, and is based on special properties of chiral operators (=operators corresponding to the lowest component of chiral superfields) in $\mathcal{N} = 1$ supersymmetric field theories in four dimensions. It is well-known that correlators of such operators do not depend on the space-time coordinates and that they factorize:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle . \tag{3.2}$$

On the set of chiral operators one considers the equivalence relation:

$$\mathcal{O}_1 \equiv \mathcal{O}_2 + c_{\dot\alpha} \bar{Q}^{\dot\alpha}(\dots) , \tag{3.3}$$

where $c_{\dot\alpha} \bar{Q}^{\dot\alpha}$ is an arbitrary linear combination of the anti-chiral supercharges. This equivalence relation is compatible with the operator product structure, and modding out the space of chiral operators by (3.3) leads to the chiral ring. Equivalence of two chiral operators under (3.3) implies equality of their vevs:

$$\langle \mathcal{O}_1 \rangle = \langle \mathcal{O}_2 \rangle \ . \tag{3.4}$$

An important relation in the chiral ring is:

$$\{\mathcal{W}_\alpha^{(r)}, \mathcal{W}_\beta^{(r)}\} \equiv 0 \,, \tag{3.5}$$

which holds for any representation r of the gauge group. This is a special case of the general relation [28]:

$$\mathcal{W}_\alpha^{(r)}.\mathcal{O}^{(r)} \equiv 0 \,, \tag{3.6}$$

which holds for an arbitrary chiral operator $\mathcal{O}^{(r)}$ transforming in the representation r of the gauge group. The dot in this equation indicates the action of $\mathcal{W}_\alpha$ in the representation r. For our theory, one finds:

$$\begin{aligned} \mathcal{W}_\alpha.\Phi &= [\mathcal{W}_\alpha, \Phi] \equiv 0 && (3.7)\\ \mathcal{W}_\alpha.A &= \mathcal{W}_\alpha A + A\mathcal{W}_\alpha^T \equiv 0 && (3.8)\\ \mathcal{W}_\alpha.S &= -S\mathcal{W}_\alpha - \mathcal{W}_\alpha^T S \equiv 0 && (3.9)\\ \mathcal{W}_\alpha.Q_f &= \mathcal{W}_\alpha Q_f \equiv 0 \,, && (3.10) \end{aligned}$$

where $\mathcal{W}_\alpha$ in the right hand sides are taken in the adjoint representation and juxtaposition stands for matrix multiplication. Note that $\mathcal{W}_\alpha$ acts on the product AS through the commutator (since this product transforms in the adjoint representation).

The generalized Konishi anomaly is the anomalous Ward identity for a local holomorphic field transformation:

$$\mathcal{O}^{(r)} \longrightarrow \mathcal{O}^{(r)} + \delta\mathcal{O}^{(r)} \,. \tag{3.11}$$

The supercurrent generator has the form $J = (\mathbf{O}^{(r)})^\dagger e^{\mathbf{V}^{(r)}} \delta\mathbf{O}^{(r)}$ where $\mathbf{V}^{(r)}$ is the vector superfield in the representation r and $\mathbf{O}^{(r)}$ is the chiral superfield associated with $\mathcal{O}^{(r)}$. The chiral ring relation induced by the generalized Konishi anomaly for this current is:

$$\delta\mathcal{O}_I \frac{\partial W}{\partial \mathcal{O}_I} \equiv -\frac{1}{32\pi^2} \mathcal{W}_I^{\alpha\ J} \mathcal{W}_{\alpha, J}^{\ \ \ K} \frac{\partial(\delta\mathcal{O}_K)}{\partial \mathcal{O}_I} \,, \tag{3.12}$$

where the capital indices enumerate a basis of the representation r. We will investigate the generalized Konishi relations corresponding to the field transformations:

$$\begin{aligned} \delta\Phi &= \frac{\mathcal{W}^\alpha \mathcal{W}_\alpha}{z - \Phi} && (3.13)\\ \delta\Phi &= \frac{1}{z-\Phi} && (3.14)\\ \delta A &= \frac{\mathcal{W}^\alpha}{z-\Phi} A \frac{(\mathcal{W}_\alpha)^T}{z - \Phi^T} && (3.15)\\ \delta A &= \frac{1}{z-\Phi} A \frac{1}{z-\Phi^T} && (3.16) \end{aligned}$$

$$\delta S = \frac{1}{z-\Phi^T} S \frac{1}{z-\Phi} \tag{3.17}$$

$$\delta Q_f = \sum_{g=1}^{N_F} \frac{\lambda_{fg}}{z-\Phi} Q_g \,. \tag{3.18}$$

In the last equation, λ is an arbitrary matrix in flavor space.

Writing $\mathcal{W}^2 = \mathcal{W}^\alpha \mathcal{W}_\alpha$, we define:

$$\mathbf{R}(z) := -\frac{1}{32\pi^2} \operatorname{tr}\left(\frac{\mathcal{W}^2}{z-\Phi}\right) \tag{3.19}$$

$$\mathbf{w}_\alpha(z) := \frac{1}{4\pi} \operatorname{tr}\left(\frac{\mathcal{W}_\alpha}{z-\Phi}\right) \tag{3.20}$$

$$\mathbf{T}(z) := \operatorname{tr}\left(\frac{1}{z-\Phi}\right) . \tag{3.21}$$

Then it is shown in appendix C that transformations (3.13)–(3.18) generate the chiral ring relations:

$$-\frac{1}{32\pi^2} \operatorname{tr}\left(\frac{W'(\Phi)\mathcal{W}^2}{z-\Phi}\right) - \frac{1}{32\pi^2} \operatorname{tr}\left(S\frac{\mathcal{W}^2}{z-\Phi}A\right) \equiv \mathbf{R}(z)^2 \tag{3.22}$$

$$\operatorname{tr}\left(\frac{W'(\Phi)}{z-\Phi}\right) + \operatorname{tr}\left(S\frac{1}{z-\Phi}A\right) \equiv 2\mathbf{R}(z)\mathbf{T}(z) + \mathbf{w}^\alpha(z)\mathbf{w}_\alpha(z) \tag{3.23}$$

$$-\frac{1}{32\pi^2} \operatorname{tr}\left(S\frac{\mathcal{W}^2}{z-\Phi}A\right) \equiv \frac{1}{2}\mathbf{R}(z)^2 \tag{3.24}$$

$$\operatorname{tr}\left(S\frac{1}{z-\Phi}A\right) \equiv \mathbf{R}(z)\mathbf{T}(z) + 2\mathbf{R}'(z) - \\ -\frac{1}{2}\mathbf{w}^\alpha(z)\mathbf{w}_\alpha(z) \tag{3.25}$$

$$\operatorname{tr}\left(S\frac{1}{z-\Phi}A\right) + \sum_f Q_f^T \frac{1}{z-\Phi^T} S \frac{1}{z-\Phi} Q_f \equiv \mathbf{R}(z)\mathbf{T}(z) - 2\mathbf{R}'(z) - \\ -\frac{1}{2}\mathbf{w}^\alpha(z)\mathbf{w}_\alpha(z) \tag{3.26}$$

$$2Q_f^T S \frac{1}{z-\Phi} Q_g \equiv \mathbf{R}(z)\delta_{fg} \,. \tag{3.27}$$

Taking the trace of the last equation gives:

$$2Q_f^T S \frac{1}{z-\Phi} Q_f \equiv \mathbf{R}(z) N_F \,. \tag{3.28}$$

We next take the vacuum expectation values of these chiral ring relations. Let us define:

$$R(z) := \langle \mathbf{R}(z) \rangle \tag{3.29}$$

$$T(z) := \langle \mathbf{T}(z) \rangle \tag{3.30}$$

$$M(z) := \left\langle \operatorname{tr}\left(S\frac{1}{z-\Phi}A\right)\right\rangle \tag{3.31}$$

$$\begin{aligned}
M_Q(z) &= \sum_f \left\langle Q_f^T \frac{1}{z-\Phi^T} S \frac{1}{z-\Phi} Q_f \right\rangle && (3.32)\\
K(z) &:= -\frac{1}{32\pi^2}\left\langle \operatorname{tr}\left(S \frac{\mathcal{W}^2}{z-\Phi} A\right)\right\rangle && (3.33)\\
L(z) &:= \sum_f \left\langle Q_f^T S \frac{1}{z-\Phi} Q_f \right\rangle . && (3.34)
\end{aligned}$$

Introducing the degree $d-1$ polynomials:

$$\begin{aligned}
f(z) &= -\frac{1}{32\pi^2}\operatorname{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\mathcal{W}^2\right) && (3.35)\\
c(z) &= \operatorname{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right), && (3.36)
\end{aligned}$$

we write:

$$\begin{aligned}
-\frac{1}{32\pi^2}\operatorname{tr}\left(\frac{W'(\Phi)\mathcal{W}^2}{z-\Phi}\right) &= W'(z)R(z) - f(z) && (3.37)\\
\operatorname{tr}\left(\frac{W'(\Phi)}{z-\Phi}\right) &= W'(z)T(z) - c(z)\,. && (3.38)
\end{aligned}$$

Noticing that the vevs of spinor fields vanish due to Lorentz invariance, we find the following Ward identities for the generating functions (3.29)–(3.34):

$$\begin{aligned}
R(z)^2 - K(z) - W'(z)R(z) + f(z) &= 0 && (3.39)\\
2R(z)T(z) - W'(z)T(z) - M(z) + c(z) &= 0 && (3.40)\\
K(z) - \frac{1}{2}R(z)^2 &= 0 && (3.41)\\
M(z) - R(z)T(z) - 2R'(z) &= 0 && (3.42)\\
M_Q(z) + M(z) - R(z)T(z) + 2R'(z) &= 0 && (3.43)\\
2L(z) - N_F R(z) &= 0\,. && (3.44)
\end{aligned}$$

It is easy to eliminate $K(z), M(z), M_Q(z)$ and $L(z)$ from these equations to find:

$$\begin{aligned}
R(z)^2 - 2W'(z)R(z) + 2f(z) &= 0 && (3.45)\\
T(z)R(z) - W'(z)T(z) - 2R'(z) + c(z) &= 0\,. && (3.46)
\end{aligned}$$

Given a solution $(R(z), T(z))$ of these constraints, the quantities K, M, M_Q and L are given by:

$$\begin{aligned}
K(z) &= \frac{1}{2}R(z)^2\\
M(z) &= 2R'(z) + R(z)T(z)\\
M_Q(z) &= -4R'(z)\\
L(z) &= \frac{N_F}{2}R(z)\,. && (3.47)
\end{aligned}$$

Hence all solutions are parameterized by the $2d$ complex coefficients of the polynomials $f(z)$ and $c(z)$.

The generalized Konishi relations involving the flavors Q_f have an interesting implication. Expanding the last two equations in (3.47) to leading order in $1/z$ gives:

$$\sum_f \langle Q_f^T S Q_f \rangle = 4\mathcal{S} \tag{3.48}$$

and

$$\sum_f \langle Q_f^T S Q_f \rangle = \frac{N_F}{2}\mathcal{S}\,, \tag{3.49}$$

where $\mathcal{S} = -\frac{1}{32\pi^2}\langle \mathrm{tr}\,\mathcal{W}^2\rangle$ is the gaugino condensate. If $\mathcal{S}$ is non-vanishing, then compatibility of these two equations requires that we set $N_F = 8$, which is also required for canceling the chiral anomaly. Any other value is incompatible with the existence of a gaugino condensate.

Observation: The particular form of (3.27) is related to the $O(N_F)$ flavor symmetry of the field theory with tree-level superpotential (2.1):

$$Q_f \to Q'_f := r_{fg} Q_g\,, \tag{3.50}$$

where r is a general complex orthogonal matrix. Since this symmetry is unbroken after confinement, the quantity $L_{fg} = \langle Q_f^T S \frac{1}{z-\Phi} Q_g\rangle$ must be $O(N_F)$ invariant and thus proportional to δ_{fg}. Equation (3.27) shows that the proportionality factor is given by $\frac{1}{2}R(z)$.

3.2 Comparison with the SO(N) model with symmetric matter

The Konishi relations (3.45) and (3.46) coincide with those of the SO(N) field theory with a single complex chiral superfield X transforming in the symmetric representation, and with a tree-level superpotential given by $\mathrm{tr}\,W(X)$. The Konishi relations for this theory were derived in [18] (see equation (18) of that reference[4]) and one immediately checks that they agree with our equations (3.45) and (3.46). More precisely, the SO(N) theory with symmetric matter admits the following resolvent-like objects:

$$R_s(z) := -\left\langle \frac{1}{32\pi^2}\,\mathrm{tr}\left(\frac{\mathcal{W}^2}{z-X}\right)\right\rangle \tag{3.51}$$

$$T_s(z) := \left\langle \mathrm{tr}\left(\frac{1}{z-X}\right)\right\rangle, \tag{3.52}$$

which were shown in [18] to obey the two equations (3.45) and (3.46). This leads to the identification:

$$\begin{aligned} R(z) &\longleftrightarrow R_s(z) \\ T(z) &\longleftrightarrow T_s(z)\,. \end{aligned} \tag{3.53}$$

which maps a solution of our Konishi constraints to a solution of the Konishi relations of the SO(N) theory (this works because equations (3.47) completely determine the other resolvent-like objects of the chiral theory given a solution (R,T) of (3.45) and (3.46)).

[4]The paper [18] considers a slight extension of our SO(N) theory, by adding some fundamental and anti-fundamental matter beyond the symmetric field. The case of interest for our purpose is recovered from the equations of [18] by setting their quark mass matrix to zero.

4. The matrix model

The general conjecture of [1] suggests that the effective superpotential of our field theory should be described by the holomorphic matrix model:[5]

$$Z = \frac{1}{|G|}\int_\Gamma d\hat{\Phi} d\hat{A} d\hat{S} d\hat{Q} e^{-\frac{\hat{N}}{g}\mathcal{S}_{mm}(\hat{\Phi},\hat{A},\hat{S},\hat{Q})}\,, \tag{4.1}$$

where $|G|$ is a normalization factor and:

$$\mathcal{S}_{mm}(\hat{\Phi},\hat{A},\hat{S},\hat{Q}) = \mathrm{tr}\left[W(\hat{\Phi}) + \hat{S}\hat{\Phi}\hat{A}\right] + \sum_{f=1}^{\hat{N}_F} \hat{Q}_f^T \hat{S}\hat{Q}_f = \mathrm{tr}\left[W(\hat{\Phi}) + \hat{\Phi}\hat{A}\hat{S} + \sum_{f=1}^{\hat{N}_F}\hat{Q}_f\hat{Q}_f^T\hat{S}\right]. \tag{4.2}$$

Here $\hat{\Phi}$ is an arbitrary complex $\hat{N}\times\hat{N}$ matrix, the complex $\hat{N}\times\hat{N}$ matrices $\hat{S}$ and $\hat{A}$ are symmetric and antisymmetric respectively and $\hat{Q}$ is a general complex $\hat{N}\times\hat{N}_F$ matrix whose columns we denote by $\hat{Q}_f$. The integration measure $d\mu = d\hat{\Phi}d\hat{A}d\hat{S}d\hat{Q} = d\hat{\Phi}d\hat{A}d\hat{S}\prod_{f=1}^{\hat{N}_F} d\hat{Q}_f$ is given by:

$$d\hat{\Phi} = \bigwedge_{i,j=1}^{\hat{N}} d\hat{\Phi}_{ij}\,, \qquad d\hat{S} = \bigwedge_{i\le j} d\hat{S}_{ij}\,, \qquad d\hat{A} = \bigwedge_{i<j} d\hat{A}_{ij}\,, \qquad d\hat{Q} = \bigwedge_{f=1}^{\hat{N}_F}\bigwedge_{i=1}^{\hat{N}} d\hat{Q}^i_f\,. \tag{4.3}$$

where $\bigwedge$ denotes the wedge product and we use the lexicographic order of indices to give unambiguous meaning to the various products of one-forms. For example, the notation $\bigwedge_{i\le j} d\hat{S}_{ij}$ means:

$$d\hat{S}_{11}\wedge d\hat{S}_{12}\wedge\cdots\wedge d\hat{S}_{1\hat{N}}\wedge d\hat{S}_{22}\wedge d\hat{S}_{23}\wedge\cdots\wedge d\hat{S}_{2\hat{N}}\wedge\cdots\wedge d\hat{S}_{\hat{N}-1\hat{N}-1}\wedge d\hat{S}_{\hat{N}-1\hat{N}}\wedge d\hat{S}_{\hat{N}\hat{N}}\,. \tag{4.4}$$

Of course, the ordering convention can be chosen arbitrarily since changing it produces an irrelevant sign prefactor in the matrix integral.

The measure $d\mu$ is a top holomorphic form on the complex space:

$$\mathcal{M} = \{(\hat{\Phi},\hat{S},\hat{A},\hat{Q}_1\ldots\hat{Q}_{\hat{N}_F})|\hat{S}^T = \hat{S}, \hat{A}^T = -\hat{A}\}\,. \tag{4.5}$$

The integral in (4.1) is performed on a boundary-less real submanifold Γ of $\mathcal{M}$ whose closure is non-compact and which is chosen such that $\dim_{\mathbb{R}}\Gamma = \dim_{\mathbb{C}}\mathcal{M}$. The model (4.1) admits the $O(\hat{N}_F)$ flavor symmetry:

$$\hat{Q}_f \to \hat{Q}'_f := r_{fg}\hat{Q}_g\,, \tag{4.6}$$

where r is an $\hat{N}_F\times\hat{N}_F$ orthogonal matrix.

[5] It is interesting to notice that this holomorphic matrix model does not have a real ('hermitean') counterpart. This is due to the fact that our matter representation is intrinsically complex.

Since anomaly cancellation in our field theory requires $N_F = 8$, one is tempted to set $\hat{N}_F = 8$ as well. It turns out that this naive identification cannot hold in our case. To understand why, notice that both the matrix model action (4.2) and the integration measure are invariant under the following $\mathrm{SL}(\hat{N}, \mathbb{C})$ gauge transformations:

$$\hat{\Phi} \to U\hat{\Phi}U^{-1}\,, \qquad \hat{S} \to (U^{-1})^T\hat{S}U^{-1}\,, \qquad \hat{A} \to U\hat{A}U^T\,, \qquad \hat{Q}_f \to U\hat{Q}_f\,, \tag{4.7}$$

where U is a complex $\hat{N} \times \hat{N}$ matrix of unit determinant (invariance of the measure is discussed in detail in appendix A). To preserve this symmetry, one must choose Γ to be stabilized by the action (4.7) of $\mathrm{SL}(\hat{N}, \mathbb{C})$.

The matrix model action is in fact invariant under the full $GL(\hat{N}, \mathbb{C})$ group acting as in (4.7). However, the measure $d\mu$ is *not* invariant under the central $\mathbb{C}^*$ subgroup of $GL(\hat{N}, \mathbb{C})$ unless $\hat{N}_F = 2$. Taking $U = \xi \mathbf{1}_{\hat{N}}$ in (4.7) with $\xi \in \mathbb{C}^*$, we have:

$$\hat{A} \to \xi^2 \hat{A}\,, \qquad \hat{S} \to \xi^{-2}\hat{S} \qquad \text{and} \qquad \hat{Q}_f \to \xi\hat{Q}_f\,, \tag{4.8}$$

which gives:

$$d\mu \to \xi^{\hat{N}(\hat{N}_F-2)} d\mu\,. \tag{4.9}$$

Let us assume $\hat{N}_F \neq 2$ and choose Γ to be $GL(\hat{N}, \mathbb{C})$ invariant. Then, since the matrix model action (4.2) is invariant under (4.8), while the measure transforms nontrivially, invariance of the integral (4.1) under coordinate transformations shows that:

$$Z = \xi^{\hat{N}(\hat{N}_F-2)} Z\,. \tag{4.10}$$

Thus Z must either vanish or equal complex infinity![6] A similar argument shows that the integral $\int d\mu F e^{-\frac{\hat{N}}{g}\mathcal{S}_{mm}}$ must vanish or be infinite for any functional $F(\hat{\Phi}, \hat{A}, \hat{S}, \{\hat{Q}_f\})$ which is invariant under (4.8). In particular, the expectation value:

$$\langle F \rangle := \frac{1}{Z}\int d\mu F e^{-\frac{\hat{N}}{g}\mathcal{S}_{mm}} \tag{4.11}$$

of any such functional is ill-defined ! This means that the matrix model predicted by a naive application of the conjecture of [1] is not well-defined.

That subtleties can arise when attempting to apply the conjecture of [1] to chiral field theories is not completely unexpected, since most derivations of this conjecture up to date have concentrated on real matter representations, which prevent the appearance of net chirality. The phenomenon we just discussed shows that one must modify the original conjecture of [1] in order to adapt it to the chiral context.

Thus we are lead to consider the matrix model with $\hat{N}_F = 2$. Then both the action (4.2) and the integration measure are invariant under $GL(\hat{N}, \mathbb{C})$ transformations of the form (4.7), where U is now an arbitrary complex invertible matrix. In Subsections 4.3 and 4.4 below, we shall show explicitly that the model with $\hat{N}_F = 2$ is well-defined by relating it to the holomorphic matrix model associated with the $\mathrm{SO}(N)$ theory with symmetric matter.

[6] The second solution is allowed since Z is complex and the point at infinity in the complex plane does satisfy $\infty = \xi^{\hat{N}(\hat{N}_F-2)}\infty$.

4.1 Loop equations

In this subsection, we extract the loop equations of the model (4.1). Although the correlation functions are not well defined unless $\hat{N}_F = 2$, we will work formally with an arbitrary value of $\hat{N}_F$. This will allow us to re-discover the constraint $\hat{N}_F = 2$ as a consistency condition between the loop equations, in a manner similar to the way in which we recovered the condition $N_F = 8$ in Subsection 3.1. by using the Konishi constraints of the field theory.

In addition to the matrix model resolvent:

$$\omega(z) = \frac{g}{\hat{N}} \operatorname{tr} \frac{1}{z - \hat{\Phi}}\,, \tag{4.12}$$

we shall consider the objects:

$$\begin{aligned} k(z) &= \frac{g}{\hat{N}} \operatorname{tr} \left[\hat{S} \frac{1}{z - \hat{\Phi}} \hat{A} \right] &(4.13)\\ m_Q(z) &= \hat{Q}_f^T \frac{1}{z - \hat{\Phi}^T} \hat{S} \frac{1}{z - \hat{\Phi}} \hat{Q}_f &(4.14)\\ l(z) &= \hat{Q}_f^T \hat{S} \frac{1}{z - \hat{\Phi}} \hat{Q}_f\,. &(4.15) \end{aligned}$$

We will show that these fulfill the loop equations:

$$\begin{aligned} \langle \omega(z)^2 - W'(z)\omega(z) - k(z) + \tilde{f}(z) \rangle &= 0 &(4.16)\\ \left\langle \frac{1}{2}\omega(z)^2 + \frac{1}{2}\frac{g}{\hat{N}}\omega'(z) - k(z) \right\rangle &= 0 &(4.17)\\ \langle \omega'(z) + m_Q(z) \rangle &= 0 &(4.18)\\ \langle \hat{N}_F \omega(z) - 2l(z) \rangle &= 0\,, &(4.19) \end{aligned}$$

where:

$$\tilde{f}(z) := \frac{g}{\hat{N}} \operatorname{tr} \frac{W'(z) - W'(\hat{\Phi})}{z - \hat{\Phi}} \tag{4.20}$$

is a polynomial of degree $d - 1$.

Before giving the derivation of these constraints, let us note that one can eliminate $\langle k(z) \rangle$ between (4.17) and (4.16) to find an equation for the resolvent:

$$\left\langle \omega(z)^2 - \frac{g}{\hat{N}}\omega'(z) - 2W'(z)\omega(z) + 2\tilde{f}(z) \right\rangle = 0\,. \tag{4.21}$$

Given a solution $\langle \omega(z) \rangle$, relation (4.17), (4.18) and (4.19) determine the averages of $k(z)$, $m_Q(z)$ and $l(z)$ as follows:

$$\begin{aligned} \langle k(z) \rangle &= \frac{1}{2}\langle \omega(z)^2 + \frac{g}{\hat{N}}\omega'(z) \rangle \\ \langle m_Q(z) \rangle &= -\langle \omega'(z) \rangle \\ \langle l(z) \rangle &= \frac{\hat{N}_F}{2}\langle \omega(z) \rangle\,. &(4.22) \end{aligned}$$

The leading order in the large z expansion of the last two equations gives:

$$\langle \hat{Q}_f^T \hat{S} \hat{Q}_f \rangle = g \tag{4.23}$$

$$\langle \hat{Q}_f^T \hat{S} \hat{Q}_f \rangle = \frac{\hat{N}_F}{2} g\,, \tag{4.24}$$

where we used the large z behavior of the resolvent:

$$\langle \omega(z) \rangle \approx \frac{g}{z} + O(\frac{1}{z^2})\,. \tag{4.25}$$

Since we of course take $g \neq 0$, equations (4.23) are consistent only if $\hat{N}_F = 2$. We now proceed to give the proof of (4.16)–(4.19).

4.2 Direct derivation of the loop equations

Consider the identity:

$$\int d\hat{\Phi}\, d\hat{A}\, d\hat{S}\, d\hat{Q}\ \frac{\partial}{\partial \hat{\Phi}_i{}^j} \left[\left(\frac{1}{z - \hat{\Phi}} \right)_i{}^j e^{-\mathcal{S}_{mm}} \right] = 0\,. \tag{4.26}$$

Using:

$$\frac{\partial}{\partial \hat{\Phi}_i{}^j} \left(\frac{1}{z - \hat{\Phi}} \right)_i{}^j = \left(\operatorname{tr} \frac{1}{z - \hat{\Phi}} \right)^2, \tag{4.27}$$

this leads to:

$$\left\langle \omega(z)^2 - \frac{g}{\hat{N}} \operatorname{tr} \left(\frac{W'(\hat{\Phi})}{z - \hat{\Phi}} \right) - k(z) \right\rangle = 0\,. \tag{4.28}$$

Equation (4.20) allows us to write (4.28) in the form (4.16).

In order to find an additional equation for $k(z)$, we consider the identity:

$$\int d\hat{\Phi}\, d\hat{A}\, d\hat{S}\, d\hat{Q}\ \frac{\partial}{\partial \hat{A}_{ij}} \left[\left(\frac{1}{z - \hat{\Phi}} \hat{A} \frac{1}{z - \hat{\Phi}^T} \right)_{ij} e^{-\mathcal{S}_{mm}} \right] = 0\,. \tag{4.29}$$

This implies equation (4.17) upon using the relations:

$$\operatorname{tr} \left(\hat{S} \frac{\hat{\Phi}}{z - \hat{\Phi}} \hat{A} \frac{1}{z - \hat{\Phi}^T} \right) = -\operatorname{tr} \left(\hat{S} \frac{1}{z - \hat{\Phi}} \hat{A} \frac{\hat{\Phi}^T}{z - \hat{\Phi}^T} \right) = \operatorname{tr} \left(\hat{S} \frac{1}{z - \hat{\Phi}} \hat{A} \right), \tag{4.30}$$

which follow trivially from the symmetry properties of $\hat{S}$ and $\hat{A}$ and invariance of the trace under transposition.[7]

We next consider the identity:

$$\int d\hat{\Phi}\, d\hat{A}\, d\hat{S}\, d\hat{Q}\ \frac{\partial}{\partial \hat{S}^{ij}} \left[\left(\frac{1}{z - \hat{\Phi}^T} \hat{S} \frac{1}{z - \hat{\Phi}} \right)^{ij} e^{-\mathcal{S}_{mm}} \right] = 0\,, \tag{4.31}$$

which gives:

$$\left\langle \frac{1}{2} \omega(z)^2 - \frac{1}{2} \frac{g}{\hat{N}} \omega'(z) - k(z) - \frac{g}{\hat{N}} m_Q(z) \right\rangle = 0\,. \tag{4.32}$$

Together with (4.17), this implies the third loop equation (4.18).

[7] For this, it is useful to notice that $\operatorname{tr}(\hat{S} \frac{1}{z - \hat{\Phi}} \hat{A} \frac{1}{z - \hat{\Phi}^T}) = 0$.

Finally, we can derive a relation involving the flavors. For this, we start with the identity:

$$\int d\hat{\Phi}\, d\hat{A}\, d\hat{S}\, d\hat{Q}\ \frac{\partial}{\partial \hat{Q}^i_f} \left[\left(\frac{\lambda_{fg}}{z-\hat{\Phi}} \right)_{ij} \hat{Q}^j_g\, e^{-\mathcal{S}_{mm}} \right] = 0\,, \tag{4.33}$$

where λ is an arbitrary matrix in flavor space.

Equation (4.33) gives:

$$\left\langle \lambda_{ff}\omega(z) - 2\sum_{g=1}^{\hat{N}_F} \hat{Q}^T_f \hat{S} \frac{\lambda_{fg}}{z-\hat{\Phi}} \hat{Q}_g \right\rangle = 0\,. \tag{4.34}$$

Since λ is arbitrary, this implies:

$$\langle \omega(z)\delta_{fg} - 2\hat{Q}^T_f \hat{S} \frac{1}{z-\hat{\Phi}} \hat{Q}_g \rangle = 0\,. \tag{4.35}$$

Setting $f=g$ and summing over f gives equation (4.19).

Since the matrix model admits the $O(\hat{N}_F)$ flavor symmetry (4.6), it follows that the expectation value of any flavor two-tensor must be $O(\hat{N}_F)$ invariant and thus:

$$\langle \hat{Q}^T_f \hat{S} \frac{1}{z-\hat{\Phi}} \hat{Q}_g \rangle = \frac{1}{\hat{N}_F} \langle l(z) \rangle \delta_{fg}\,. \tag{4.36}$$

Combining with (4.19), this gives:

$$\langle \hat{Q}^T_f \hat{S} \frac{1}{z-\hat{\Phi}} \hat{Q}_g \rangle = \frac{1}{2} \langle \omega(z) \rangle \delta_{fg}\,, \tag{4.37}$$

which is the matrix model analogue of equation (3.27).

4.3 The eigenvalue representation

As mentioned above, the integration in (4.1) should be performed over an appropriate multidimensional contour inside the space of complex matrices $\hat{\Phi}$, $\hat{S}$, $\hat{A}$ and $\hat{Q}_f$. We shall specify this contour by first fixing the gauge through diagonalizing[8] $\hat{\Phi}$ and imposing conditions on the remaining matrices *after* gauge-fixing. This procedure is clearly gauge-invariant in the sense that it defines a multidimensional contour which is stabilized by the action of the complex gauge group $GL(\hat{N}, \mathbb{C})$. Using the gauge symmetry to bring $\hat{\Phi}$ to the form:

$$\hat{\Phi} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\hat{N}}) \tag{4.38}$$

allows us to write the matrix model action (4.2) as:

$$\mathcal{S}_{mm} = \sum_{i=1}^{\hat{N}} W(\lambda_i) + \sum_{i<j} \left[(\lambda_i - \lambda_j)\hat{A}_{ij} + 2\sum_{f=1}^{\hat{N}_F} \hat{Q}^i_f \hat{Q}^j_f \right] \hat{S}_{ij} + \sum_{i=1}^{\hat{N}} \left[\sum_{f=1}^{\hat{N}_F} (\hat{Q}^i_f)^2 \right] \hat{S}_{ii}\,. \tag{4.39}$$

[8]The fact that Φ is diagonalizable is part of the definition of our multidimensional contour. As in [4], this implements a point-splitting regularization of the matrix integral, which can be removed trivially because the resulting Vandermonde determinant (to be described below) vanishes for coinciding eigenvalues.

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To make sense of the remaining integral, we impose the conditions:

$$\lambda_i \in \gamma\,, \qquad \hat{S}_{ij} \in i\mathbb{R}\,, \qquad \hat{A}_{ij} \in \mathbb{R}\,, \qquad \hat{Q}^i_f \in \mathbb{R}\,, \tag{4.40}$$

where γ is an open contour in the complex plane whose asymptotic behavior is dictated by the highest degree term in W as explained in [4] (one can take γ to coincide with the real axis if and only if W is a polynomial of even degree). This amounts to choosing:

$$\begin{aligned}\Gamma = \Big\{ &(\hat{\Phi}, \hat{S}, \hat{A}, \hat{Q}_1 \dots \hat{Q}_{\hat{N}_F}) \in \mathcal{M} |\ \exists U \in GL(\hat{N}, \mathbb{C}) \quad \text{such that} \\ &U\hat{\Phi}U^{-1} = \mathrm{diag}(\lambda_1 \dots \lambda_{\hat{N}}) \quad \text{with} \lambda_1 \dots \lambda_{\hat{N}} \in \gamma \quad \text{and} \\ &U^{-T}\hat{S}U^{-1} \in \mathrm{Mat}(\hat{N}, \hat{N}, \mathbb{R}),\ U\hat{A}U^T \in \mathrm{Mat}(\hat{N}, \hat{N}, \mathbb{R}), \\ &U\hat{Q}_f \in \mathrm{Mat}(\hat{N}, 1, \mathbb{R}) \quad \text{for all } f \Big\}\,.\end{aligned} \tag{4.41}$$

With this choice of integration contour, we obtain:

$$Z = (2\pi i)^{\hat{N}(\hat{N}+1)/2} Z_Q Z_{\text{red}}\,, \tag{4.42}$$

where:

$$Z_Q = \int_{\mathbb{R}^{\hat{N}_F \hat{N}}} d\hat{Q} \prod_{j=1}^{\hat{N}} \delta\left(\sum_{f=1}^{\hat{N}_F} (\hat{Q}^j_f)^2\right) \tag{4.43}$$

and:

$$Z_{red} = \int_\gamma d\lambda_1 \dots \int_\gamma d\lambda_{\hat{N}} \prod_{i<j} (\lambda_i - \lambda_j) e^{-\frac{\hat{N}}{g}\sum_{i=1}^{\hat{N}} W(\lambda_i)}\,. \tag{4.44}$$

To arrive at (4.42), we performed the integrals over $\hat{S}_{ij}$, which appear linearly in the action. This gives a product of delta-functions which allows us to reduce the integral to (4.42). From this expression, it is clear that the variables $\hat{A}$, $\hat{S}$ and $\hat{Q}_f$ decouple from $\hat{\Phi}$. The interesting dynamics of the model is contained in the reduced partition function (4.44), which differs from that of a usual (adjoint) one-matrix model only because the Vandermonde determinant $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)$ is *not* squared in (4.44). As we shall see below, Z_{red} can in fact be identified with the partition function of a holomorphic $\mathrm{SO}(\hat{N}, \mathbb{C})$ — invariant one matrix model with a single symmetric field $\hat{X}$ and action $\mathrm{tr}\, W(\hat{X})$. This, of course, is just the holomorphic matrix model associated with the $\mathrm{SO}(N)$ theory with symmetric matter via the Dijkgraaf-Vafa conjecture.

Observation: The integral (4.43) is finite and non-vanishing precisely in the case of interest $\hat{N}_F = 2$. In this case, one easily checks that:

$$Z_Q = \pi^{\hat{N}}\,. \tag{4.45}$$

For $\hat{N}_F > 2$ we find $Z_Q = 0$, while for $\hat{N}_F = 1$ we have $Z_Q = \infty$. This agrees with our previous discussion.

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4.4 Relation to the matrix model of the SO(N) theory

As mentioned above, it turns out that the reduced model described by (4.44) agrees with the matrix model associated with the SO(N) theory with symmetric matter via the Dijkgraaf-Vafa correspondence. The hermitean version of the latter is defined through:

$$Z_s := \frac{1}{|G_s|}\int d\hat{X} e^{-\frac{\hat{N}}{g}\operatorname{tr} W(\hat{X})}\,, \tag{4.46}$$

where $|G_s| = \mathrm{vol}(\mathrm{SO}(\hat{N},\mathbb{R})/S_{\hat{N}})$ and $\hat{X}$ is a real symmetric $\hat{N}\times\hat{N}$ matrix (thus $\hat{X}^T = \hat{X}$ and $\hat{X}^\dagger = \hat{X}$). In the hermitean case, the measure is given by:

$$d\hat{X} = \prod_{i\leq j} d\hat{X}_{ij}\,. \tag{4.47}$$

The partition function (4.46) and the measure (4.47) are invariant under the transformations:

$$\hat{X} \to V\hat{X}V^T\,, \tag{4.48}$$

where V is an element of $\mathrm{SO}(\hat{N},\mathbb{R})$.

One way to see the aforementioned correspondence is by relating the loop equations of the two models. The loop equation for the model (4.46) was extracted in [18], and involves only the resolvent:

$$\omega_s(z) = \frac{g}{\hat{N}}\operatorname{tr}\frac{1}{z-\hat{X}}\,. \tag{4.49}$$

This loop equation takes the form:

$$\left\langle \omega_s(z)^2 - \frac{g}{\hat{N}}\omega_s'(z) - 2W'(z)\omega_s(z) + 2\tilde{f}_s(z)\right\rangle_s = 0\,, \tag{4.50}$$

where $\langle\ldots\rangle_s$ denotes averages computed in the model (4.46) and $\tilde{f}_s(z)$ is the random polynomial:

$$\tilde{f}_s(z) := \frac{g}{\hat{N}}\operatorname{tr}\frac{W'(z)-W'(\hat{X})}{z-\hat{X}}\,. \tag{4.51}$$

Relation (4.50) corresponds to our equation (4.21) under the identifications:

$$\omega(z) \longleftrightarrow \omega_s(z) \tag{4.52}$$

$$\tilde{f}(z) \longleftrightarrow \tilde{f}_s(z)\,. \tag{4.53}$$

Since the quantities $k(z), m_Q(z), l(z)$ are determined by a solution of (4.21) via equations (4.22), this gives a one to one correspondence between solutions of the two models' loop equations.

A more direct relation between the two models can be extracted from their eigenvalue representations. To see this, we must first discuss the eigenvalue representation of (4.46).

In the context of the Dijkgraaf-Vafa correspondence, one must use the holomorphic [4] version of (4.46). This is obtained by allowing $\hat{X}$ to be a complex symmetric matrix (thus removing the hermiticity constraint $\hat{X}^\dagger = \hat{X} \Longleftrightarrow \bar{\hat{X}} = \hat{X}$) and considering gauge transformations of the form (4.48), where now V is an invertible complex-valued matrix subject to the constraints $V^T = V^{-1}$ and $\det V = 1$. This amounts to working with the complexified gauge group $\mathrm{SO}(\hat{N}, \mathbb{C})$ (then the normalization prefactor $|G_s|$ is also modified as explained in [4]). In that case, (4.47) becomes the natural top holomorphic form on the complex space $\mathcal{N} = \{\hat{X} \in \mathrm{Mat}(N, \mathbb{C}) | \hat{X}^T = \hat{X}\}$ and the integral in (4.46) must be performed along a real, boundary-less submanifold Δ of this space whose dimension equals the complex dimension of $\mathcal{N}$ and whose closure is non-compact. The admissible choices of Δ are constrained by the requirement that Δ be stabilized by the action (4.48) of the complexified gauge group and that the integral (4.46) converge when calculated along Δ. This constrains the choice of Δ in terms of the leading coefficient of W [4].

To see the relation to (4.44) explicitly, it suffices to notice that integrating out the angular variables in (4.46) leads to:

$$Z_s = \int_\gamma d\lambda_1, \ldots, \int_\gamma d\lambda_{\hat{N}} \prod_{i<j} (\lambda_i - \lambda_j) e^{-\frac{\hat{N}}{g} \sum_{i=1}^{\hat{N}} W(\lambda_i)}, \tag{4.54}$$

where γ is a suitable open contour in the complex plane (since γ is constrained only by the leading term of W [4], it can be chosen to coincide with the contour used in (4.44)). The form (4.54) corresponds to choosing $\Delta := \{\hat{X} \in \mathcal{N} | \ \exists V \in \mathrm{SO}(\hat{N}, \mathbb{C})$ such that $V\hat{X}V^T = \mathrm{diag}(\lambda_1 \ldots \lambda_{\hat{N}})$ with $\lambda_1 \ldots \lambda_{\hat{N}} \in \gamma\}$. Note that the real dimension of Δ equals the complex dimension of $\mathcal{N}$, as required. Indeed, a generic complex symmetric matrix $\hat{X}$ can be diagonalized by a complex orthogonal transformation $V \in \mathrm{SO}(\hat{N}, \mathbb{C})$:

$$V\hat{X}V^T = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\hat{N}}) \tag{4.55}$$

with complex λ_j. Since this is true outside a complex codimension one locus in $\mathcal{N}$, it follows that imposing the constraints $\lambda_j \in \gamma$ simply halves the number of real parameters in $\hat{X}$.

The factor $\prod_{i<j} (\lambda_i - \lambda_j)$ is just the square root of the factor $\prod_{i<j} (\lambda_i - \lambda_j)^2$ familiar from the case of the adjoint representation of $GL(\hat{N}, \mathbb{C})$. This can be seen most easily in the holomorphic matrix model set-up by repeating the argument given in appendix A of [4] with the observation that the number of integration variables in (4.47) is reduced with respect to the case treated there due to the condition $X_{ij} = X_{ji}$. Obviously (4.54) coincides with the reduced matrix integral (4.44):

$$Z_{red} = Z_s\,. \tag{4.56}$$

Using relation (4.42), we find that the partition function of (4.1) can be written as:

$$Z = (2\pi i)^{\hat{N}(\hat{N}+1)/2} Z_s Z_Q\,, \tag{4.57}$$

where Z_Q is the quantity defined in equation (4.43). The factorization (4.57) shows that the average in our model of any functional $F(\hat{\Phi})$ which does not depend on $\hat{S}$, $\hat{A}$ or $\hat{Q}_f$ coincides with the average of $F(\hat{X})$ in the $\mathrm{SO}(\hat{N})$ model (4.46):

$$\langle F(\hat{\Phi}) \rangle = \langle F(\hat{X}) \rangle_s\,. \tag{4.58}$$

This explains the identifications (4.52) and gives the precise relation between the two models.

4.5 The resolvent loop equation from the eigenvalue representation

To derive the loop equation of (4.44) (or, equivalently, of (4.46)), one can follow standard procedure by starting with the identity:

$$\int \prod_{i=1}^{\hat N} d\lambda_i \sum_{i=1}^{\hat N} \frac{\partial}{\partial \lambda_i}\left[\frac{1}{z-\lambda_i}\prod_{p<q}(\lambda_p-\lambda_q)e^{-\frac{\hat N}{g}\sum_{r=1}^{\hat N} W(\lambda_r)}\right] = 0\,. \tag{4.59}$$

Performing the partial derivative gives the relation:

$$\left\langle \sum_{i=1}^{\hat N}\frac{1}{(z-\lambda_i)^2} - \frac{\hat N}{g}\sum_{i=1}^{\hat N}\frac{W'(\lambda_i)}{z-\lambda_i} + \sum_{i\neq j}\frac{1}{\lambda_i-\lambda_j}\frac{1}{z-\lambda_i}\right\rangle = 0\,. \tag{4.60}$$

Introducing the traced resolvent (4.12) and using the identity:

$$\sum_{i\neq j}\frac{1}{\lambda_i-\lambda_j}\frac{1}{z-\lambda_i} = \frac{1}{2}\left[\sum_{i,j}\frac{1}{z-\lambda_j}\frac{1}{z-\lambda_i} - \sum_k \frac{1}{(z-\lambda_k)^2}\right] \tag{4.61}$$

allows us to write (4.60) in the form:

$$\left\langle \omega(z)^2 - \frac{g}{\hat N}\omega'(z) - \frac{2g}{\hat N}\sum_{i=1}^{\hat N}\frac{W'(\lambda_i)}{z-\lambda_i}\right\rangle = 0\,. \tag{4.62}$$

Using the degree $d-1$ polynomial (4.20):

$$\tilde f(z) = \frac{g}{\hat N}\sum_{i=1}^{\hat N}\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i} \tag{4.63}$$

leads to the loop equation in the form:

$$\left\langle \omega(z)^2 - \frac{g}{\hat N}\omega'(z) - 2W'(z)\omega(z) + 2\tilde f(z)\right\rangle = 0\,, \tag{4.64}$$

which recovers (4.21).

Consider the spectral density $\rho(\lambda) = \frac{1}{\hat N}\sum_{i=1}^{\hat N}\delta(\lambda-\lambda_i)$ and the expansions:

$$\langle\omega(z)\rangle = \sum_{j\geq 0}\left(\frac{g}{\hat N}\right)^j \omega_j(z) \tag{4.65}$$

$$\langle\rho(\lambda)\rangle = \sum_{j\geq 0}\left(\frac{g}{\hat N}\right)^j \rho_j(\lambda) \tag{4.66}$$

$$\langle\tilde f(z)\rangle = \sum_{j\geq 0}\left(\frac{g}{\hat N}\right)^j \tilde f_j(z)\,. \tag{4.67}$$

In the large $\hat N$ limit, we have:

$$\omega_0(z) := \lim_{\hat N\to\infty}\langle\omega(z)\rangle = g\int d\lambda \frac{\rho_0(\lambda)}{z-\lambda}$$

$$\tilde{f}_0(z) := \lim_{\hat{N}\to\infty} \langle \tilde{f}(z) \rangle = g \int d\lambda \rho_0(\lambda) \frac{W'(z) - W'(\lambda)}{z - \lambda} \,. \tag{4.68}$$

In this limit, equation (4.62) becomes:

$$\omega_0(z)^2 - 2W'(z)\omega_0(z) + 2\tilde{f}_0(z) = 0 \,, \tag{4.69}$$

which shows that the quantity $u_0(z) = \omega_0(z) - W'(z)$ is a branch of the hyperelliptic Riemann surface:

$$u^2 = W'(z)^2 - 2\tilde{f}_0(z) \,. \tag{4.70}$$

The loop equation (4.62) can also be written in the form:

$$\left\langle \omega(z)^2 - \frac{g}{\hat{N}}\omega'(z) - 2\oint_{\mathcal{C}} \frac{dx}{2\pi i} \frac{W'(x)\omega(x)}{z - x} \right\rangle = 0 \tag{4.71}$$

where $\mathcal{C}$ is a counterclockwise contour encircling all of the eigenvalues λ_i of $\hat{\Phi}$ but not the point z. The form (4.71) can be used to give an iterative solution for the coefficients $\omega_j(z)$ of the large $\hat{N}$ expansion (4.65). In this paper, we are interested in the $\mathbb{CP}^1$ and $\mathbb{RP}^2$ diagram contributions $\omega_0(z)$ and $\omega_1(z)$. Expanding (4.71) to order $O(g/\hat{N})$ and using the relation:

$$\langle \omega(z)^2 \rangle = \langle \omega(z) \rangle^2 + O\left(\left(\frac{g}{\hat{N}}\right)^2\right) = \omega_0(z)^2 + \frac{2g}{\hat{N}}\omega_0(z)\omega_1(z) + O\left(\left(\frac{g}{\hat{N}}\right)^2\right), \tag{4.72}$$

we find:

$$\omega_0(z)^2 - 2\oint_{\mathcal{C}} \frac{dx}{2\pi i} \frac{W'(x)\omega_0(x)}{z - x} = 0 \tag{4.73}$$

$$2\omega_0(x)\omega_1(x) - 2\oint_{\mathcal{C}} \frac{dx}{2\pi i} \frac{W'(x)\omega_1(x)}{z - x} = \omega_0'(x) \,. \tag{4.74}$$

The first relation is equivalent with the large $\hat{N}$ limit (4.69) of the loop equation and determines $\omega_0(z)$ in terms of the polynomial $\tilde{f}_0(z)$. The second relation is an inhomogeneous integral equation for $\omega_1(z)$, which constraints this quantity once $\tilde{f}_0(z)$ (and thus $\omega_0(z)$) has been fixed.

5. Relation between the matrix model and field theory

5.1 Comparison of loop equations and Konishi constraints

In this subsection we find a map from matrix model to field theory quantities which takes the loop equations (4.21) and (4.22) into the Konishi constraints (3.45), (3.46) and (3.47). We set $\hat{N}_F = 2$ and $N_F = 8$ from now on.

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We start by considering a complex[9] microcanonical ensemble for our holomorphic matrix model following the procedure of [4]. This is obtained by introducing complex chemical potentials μ_k for a partition of the complex plane into domains D_k with smooth boundary, followed by a Legendre transform of the resulting grand-canonical generating function $\mathcal{F}(t,\mu) = -(\hat{N}/g)^2 \ln \mathcal{Z}(t,\mu)$, which replaces the chemical potentials by complex variables S_k[10] (remember that t_j/j are the coefficients of W). This produces the desired free energy (=microcanonical generating function) $F(t,S)$, which satisfies the equations:

$$\frac{\partial F}{\partial S_k} = \mu_k \,. \tag{5.1}$$

Working with this microcanonical ensemble amounts to imposing the constraints:

$$\langle f_k \rangle = S_k \tag{5.2}$$

where f_k is the filling fraction of the domain D_k, which we define through:

$$f_k = \oint_{\Gamma_k} \frac{dz}{2\pi i}\omega(z) \,. \tag{5.3}$$

Here Γ_k is the boundary of D_k. Note that the variables S_k must satisfy:

$$\sum_k S_k = g \,, \tag{5.4}$$

so this a 'constrained' microcanonical ensemble, as discussed in more detail in [4]. Thus passing to the microcanonical ensemble allows us to eliminate g in terms of S_i, if we choose to treat S_i as independent variables.

Expanding (5.2) to $O(g/\hat{N})$ inclusively gives:

$$\oint_{\Gamma_k} \frac{dz}{2\pi i}\omega_0(z) = S_k \tag{5.5}$$

and:

$$\oint_{\Gamma_k} \frac{dz}{2\pi i}\omega_1(z) = 0 \,. \tag{5.6}$$

Imposing these conditions fixes all coefficients of $\tilde{f}_0(z)$ and $\tilde{f}_1(z)$. In fact, condition (5.5) singles out one solution ω_0 of (4.73) while (5.6) specifies the associated solution of (4.74) by selecting the trivial solution of the associated linear homogeneous equation. Then a simple counting argument along the lines of [21] shows that that $\langle \omega(z) \rangle$ is completely determined as a function of t_j and S_k. This also determines the expectation values of $k(z), m_Q(z)$ and $l(z)$ via equations (4.22).

Let us now consider the Konishi constraints (3.45) and (3.46) of the field theory, which determine all relevant quantities through equations (3.47). To specify uniquely a solution (R,T) of (3.45) and (3.46), we shall impose the constraints:

$$\oint_{\Gamma_k} \frac{dz}{2\pi i}R(z) = \mathcal{S}_k \,, \qquad \oint_{\Gamma_k} \frac{dz}{2\pi i}T(z) = N_k \,. \tag{5.7}$$

[9] Since our matrix model does not have a real counterpart, the set-up of [4] is essential in our case.

[10] These should not be confused with the eigenvalues of the symmetric matrix S!

Note that we must impose *two conditions* on the solutions of (3.45), (3.46), which amounts to fixing the coefficients of both the polynomials f and c. As in [7], the second equation in (5.7) can be viewed as a rigorous quantum definition of the rank of the k-th factor of the unbroken low energy gauge group. This interpretation requires that we choose Γ_k such that they separate the critical points of W, which we shall assume from now on.

To map a solution of the matrix loop equations to a solution of the Konishi constraints, we shall identify:

$$\mathcal{S}_k = S_k \,. \tag{5.8}$$

Using (5.4), we find that (5.8) fixes the value of the matrix coupling constant in terms of field theory data:

$$g = \sum_k \mathcal{S}_k \,. \tag{5.9}$$

Consider the large $\hat{N}$ expansions:

$$\langle \omega(z) \rangle = \sum_{j=0}^{\infty} \left(\frac{g}{\hat{N}}\right)^j \omega_j(z) \,, \tag{5.10}$$

$$\langle k(z) \rangle = \sum_{j=0}^{\infty} \left(\frac{g}{\hat{N}}\right)^j m_j(z) \,, \tag{5.11}$$

$$\langle m_Q(z) \rangle = \sum_{j=0}^{\infty} \left(\frac{g}{\hat{N}}\right)^j m_{Qj}(z) \,, \tag{5.12}$$

$$\langle l(z) \rangle = \sum_{j=0}^{\infty} \left(\frac{g}{\hat{N}}\right)^j l_j(z) \,. \tag{5.13}$$

Expanding (4.21) to leading order, we find:

$$\omega_0(z)^2 - 2W'(z)\omega_0(z) + 2\tilde{f}_0(z) = 0 \,. \tag{5.14}$$

We also expand (4.22) to leading order and order $g/\hat{N}$ to obtain:

$$\begin{aligned} k_0(z) &= \frac{1}{2}\omega_0(z)^2 \\ m_{Q0}(z) &= -\omega_0'(z) \\ l_0(z) &= \omega_0(z) \,. \end{aligned} \tag{5.15}$$

Comparing (5.14) with (3.45) shows that $\omega_0(z)$ and $\tilde{f}_0(z)$ should be identified with $R(z)$ and $f(z)$ respectively. Moreover, equations (5.15) agree with the first and the last two equations in (3.47) provided that we identify $K(z)$ with $k_0(z)$ as well as $M_Q(z)$ with $4m_{Q0}(z)$ and $L(z)$ with $4l_0(z)$.

To recover (3.46) and the second equation in (3.47), we consider the $g/\hat{N}$ terms of (4.21) and of the first equation in (4.22), which read:

$$2\omega_0(z)\omega_1(z) - \omega_0'(z) - 2W'(z)\omega_1(z) + 2\tilde{f}_1(z) = 0 \tag{5.16}$$

and:

$$k_1(z) = \omega_0(z)\omega_1(z) + \frac{1}{2}\omega_0'(z)\,. \tag{5.17}$$

Consider the operator:

$$\delta := \sum_k N_k \frac{\partial}{\partial S_k}\,. \tag{5.18}$$

Applying this to both sides of relation (5.14) and of the first equation in (5.15) gives:

$$2\omega_0(z)\delta\omega_0(z) - 2W'(z)\delta\omega_0(z) + 2\delta\tilde{f}_0(z) = 0 \tag{5.19}$$

and:

$$\delta k_0(z) = \omega_0(z)\delta\omega_0(z)\,. \tag{5.20}$$

Combining these two equations with (5.16) and (5.17) leads to the relations:

$$\omega_0(z)[\delta\omega_0(z) + 4\omega_1(z)] - W'(z)[\delta\omega_0(z) + 4\omega_1(z)] - 2\omega_0'(z) + [\delta\tilde{f}_0(z) + 4\tilde{f}_1(z)] = 0 \tag{5.21}$$

and:

$$\delta k_0(z) + 4k_1(z) = 2\omega_0'(z) + \omega_0(z)[\delta\omega_0(z) + 4\omega_1(z)]\,. \tag{5.22}$$

These relations agree with (3.46) and (3.47) provided that we identify $T(z)$ with $\delta\omega_0(z) + 4\omega_1(z)$ as well as $M(z)$ with $\delta k_0(z) + 4k_1(z)$ and $c(z)$ with $\delta\tilde{f}_0(z) + 4\tilde{f}_1(z)$.

In conclusion, matrix model and field theory quantities must be identified according to table 1.

This correspondence recovers that used in [18] upon applying the field theory and matrix model relations (3.53) and (4.52), which connect our model with the SO(N) theory.

Matrix Model	Field Theory
S_k	$\mathcal{S}_k$
$\omega_0(z)$	$R(z)$
$\delta\omega_0(z) + 4\omega_1(z)$	$T(z)$
$\delta k_0(z) + 4k_1(z)$	$M(z)$
$k_0(z)$	$K(z)$
$4m_{Q0}(z)$	$M_Q(z)$
$4l_0(z)$	$L(z)$
$\tilde{f}_0(z)$	$f(z)$
$\delta\tilde{f}_0(z) + 4\tilde{f}_1(z)$	$c(z)$

Table 1: Identification between field theory and matrix model quantities.

5.2 The effective superpotential

The identifications of the previous subsection allow us to determine the field theory effective superpotential up to a term independent of the coefficients t_j of W. This contribution can be identified independently by using the Veneziano-Yankielowicz computation of section 2.3.

For this, we note the relation $\langle \mathrm{tr}\,\Phi^j\rangle = j\frac{\partial F}{\partial t_j}$, which implies:

$$\langle\omega(z)\rangle = \frac{d}{dW(z)}F \tag{5.23}$$

where $\frac{d}{dW(z)} := \sum_{j\geq 0}\frac{j}{z^{j+1}}\frac{\partial}{\partial t_j}$. Combing this with the expansion:

$$F = \sum_{j\geq 0}\left(\frac{g}{\hat{N}}\right)^j F_j \tag{5.24}$$

gives:

$$\omega_0(z) = \frac{d}{dW(z)} F_0 \,, \qquad \omega_1(z) = \frac{d}{dW(z)} F_1 \,. \tag{5.25}$$

On the other hand, one has the obvious field theory relation:

$$\frac{\partial W_{\rm eff}}{\partial t_j} = \frac{1}{j} \langle {\rm tr}\, \Phi^j \rangle \tag{5.26}$$

which gives:

$$T(z) = \frac{d}{dW(z)} W_{\rm eff} \,. \tag{5.27}$$

Using the identification $T(z) = \delta\omega_0(z) + 4\omega_1(z)$ (where δ is the operator given in (5.18)), this implies:

$$W_{\rm eff}(t, \mathcal{S}) = \left[\sum_i N_i \frac{\partial F_0}{\partial S_i} + 4F_1 + \psi(S) \right]_{S_i = \mathcal{S}_i} \,. \tag{5.28}$$

Here ψ is a function which depends only $\mathcal{S}_i$ but not on the coefficients of W[11]. Since we always set $S_i = \mathcal{S}_i$, we shall only use the notation $\mathcal{S}_i$ from now on. Notice that we have derived (5.28) without having to postulate some analytic continuation which would avoid identifying integer or real quantities with complex numbers. This is because the use the formalism of [4], which automatically avoids such problems. Also notice the prefactor of 4 in front of the $\mathbb{RP}^2$ contribution F_1, which arises naturally in our derivation. The fact that diagrams of topology $\mathbb{RP}^2$ generally contribute with a factor 4 to the effective superpotential was previously discussed in [46] by using the perturbative superfield approach of [5].

Expression (5.28) determines the effective superpotential only up to the coupling-independent term $\psi(\mathcal{S})$. Together with the contribution to (5.28) from the non-perturbative part of F, this term should correspond to the Veneziano-Yankielowicz potential computed in Subsection 2.3, which cannot be determined through Konishi anomaly arguments. Applying the conjecture of [1, 47] to our model leads to the proposal:

$$\psi(\mathcal{S}) = \alpha \sum_{i=1}^{d} \mathcal{S}_i \,, \tag{5.29}$$

where:

$$\alpha = (N - 4) \ln \Lambda \tag{5.30}$$

with Λ the field theory scale of Subsection 2.3. To check this expression, we now proceed to compute the non-perturbative contribution to $W_{\rm eff}$ in the matrix model and compare with the results of Subsection 2.3. This will allow us to complete the proof of the relation:

$$W_{\rm eff} = \sum_i N_i \frac{\partial F_0}{\partial \mathcal{S}_i} + 4F_1 + \alpha \sum_{i=1}^{d} \mathcal{S}_i \,. \tag{5.31}$$

[11]Remember that relation (5.9) eliminates g in terms of $\mathcal{S}_i$.

5.3 Computation of the Veneziano-Yankielowicz superpotential from the matrix model

In this subsection we show how the Veneziano-Yankielowicz contribution to the effective superpotential can be extracted from the matrix model. We shall follow the approach of [47, 7], by computing the non-perturbative contribution to the matrix integral and checking agreement between the non-perturbative part of (5.31) and the result (2.26) of Subsection 2.3.

Let us consider the classical matrix vacuum:

$$\langle\hat{\Phi}\rangle = \mathrm{diag}(\lambda_1 \mathbf{1}_{\hat{N}_1}, \ldots, \lambda_d \mathbf{1}_{\hat{N}_d}) , \qquad \langle\hat{S}\rangle = \langle\hat{A}\rangle = \langle\hat{Q}_f\rangle = 0 . \tag{5.32}$$

where λ_j are the critical points of W. Following [47, 7], we shall compute the gaussian approximation to the matrix integral expanded around this vacuum. This is the semiclassical approximation in the background (5.32).

Since we wish to compare with field theory, we must work in the microcanonical ensemble, which constraints g through relation (5.9). In the semiclassical approximation about the background (5.32), one has $\mathcal{S}_i = \hat{N}_i$ by equations (5.2) and (5.3). Then (5.9) implies $g = \hat{N}$, which means that the prefactor of the action in the exponential of the matrix integrand must be set to one. In particular, the large $\hat{N}$ expansion can be reorganized in powers of $1/\hat{N}$ rather than $g/\hat{N}$. For simplicity, we can therefore start with:

$$Z = e^{-F} = \frac{1}{\mathrm{vol}(G)} \int d\hat{\Phi} d\hat{A} d\hat{S} d\hat{Q} e^{-\mathrm{tr}[W(\hat{\Phi}) + \hat{S}\hat{\Phi}\hat{A}] - \sum_{f=1}^{2} \hat{Q}_f^T \hat{S} \hat{Q}_f} \tag{5.33}$$

and impose the microcanonical ensemble conditions $\mathcal{S}_i = N_i$ after performing the semiclassical approximation. We have set $\hat{N}_F = 2$, since this is the only case when the matrix model is well-defined.

As in Subsections 4.3 and 4.4, the partition function (5.33) can be reduced to:

$$Z = (2\pi i)^{\hat{N}(\hat{N}+1)/2} \pi^{\hat{N}} Z_s , \tag{5.34}$$

where Z_s is the partition function of the matrix model with $\mathrm{SO}(\hat{N}, \mathbb{C})$ gauge group and a complex matrix $\hat{X}$ in the symmetric representation.

It is clear from (5.34) that we can expand Z around this vacuum by expanding Z_s around the background $\langle\hat{X}\rangle = \langle\hat{\Phi}\rangle$. We let $x := \hat{X} - \langle\hat{X}\rangle$ denote the fluctuations of $\hat{X}$ and decompose x into $N_i \times N_j$ blocks x_{ij}. In the reduced model Z_s, we have an $\mathrm{SO}(\hat{N}, \mathbb{C})$ gauge symmetry which allows us to set the off-diagonal blocks of x to zero. Thus we can choose the gauge:

$$x_{ij} = 0 \qquad \text{for } i \neq j . \tag{5.35}$$

To implement this in the BRST formalism, we introduce ghosts C_{ij}, antighosts $\bar{C}_{ij}$ and Lagrange multipliers B_{ij}. These transform in the adjoint representation of $\mathrm{SO}(\hat{N}, \mathbb{C})$, so $C_{ij}^T = -C_{ji}$, $\bar{C}_{ij}^T = -\bar{C}_{ji}$ and $B_{ij}^T = -B_{ji}$. The quadratic part of the gauge-fixing action is:

$$S_{g.f.} = \sum_{i<j}^{d} \mathrm{tr}(B_{ij} x_{ij} + i(\lambda_i - \lambda_j) \bar{C}_{ij} C_{ij}) , \tag{5.36}$$

where we used the BRST transformations:

$$s\hat{X} = i[C, X]\,, \qquad sC = iC^2\,, \qquad s\bar{C} = B\,, \qquad sB = 0\,. \tag{5.37}$$

When expanding to second order in x, the diagonal blocks $x_i := x_{ii}$ acquire masses $m_i = W''(\lambda_i)$. Hence the non-perturbative piece of the partition function (5.33) is given by:

$$Z_{np} = \frac{(2\pi i)^{\hat{N}(\hat{N}+1)}\pi^{\hat{N}}}{\mathrm{Vol}(G')} \int \prod_{i=1}^{d} dx_{ii} e^{-\frac{W''(\lambda_i)}{2}\mathrm{tr}(x_i^2)} \int \prod_{j<k} dB_{jk} dC_{jk} d\bar{C}_{jk} e^{-S_{g.f.}}\,. \tag{5.38}$$

Here $G' = \prod_{i=1}^{d} \mathrm{SO}(\hat{N}_i)$ is the unbroken gauge group. Since the action is quadratic, we can choose the integration contour:

$$B_{ij} \in i\mathbb{R} \qquad x_{ii} \in \mathbb{R}\,, \tag{5.39}$$

which gives:

$$Z_{np} = \prod_{i<j} [(2\pi)^{\hat{N}_i\hat{N}_j}(\lambda_i - \lambda_j)^{\hat{N}_i\hat{N}_j}] \prod_{i=1}^{d} \left[\frac{(2\pi i)^{\frac{\hat{N}_i(\hat{N}_i+1)}{2}}\pi^{\hat{N}_i}}{\mathrm{vol}(\mathrm{SO}(\hat{N}_i))} \left(\frac{\pi}{W''(\lambda_i)}\right)^{\frac{\hat{N}_i(\hat{N}_i+1)}{4}}\right]. \tag{5.40}$$

Using [20]:

$$\log[\mathrm{vol}(\mathrm{SO}(\hat{N}))] = -\frac{\hat{N}^2}{4}\log\frac{\hat{N}}{2\pi e^{3/2}} + \frac{\hat{N}}{4}\log\frac{2\hat{N}}{\pi e} + O(\hat{N}^0)\,, \tag{5.41}$$

we extract the free energy $F_{np} = -\log Z_{np}$:

$$\begin{aligned} F_{np} = &\sum_{i=1}^{d}\left[-\left(\frac{\hat{N}_i^2}{4} + \frac{\hat{N}_i}{4}\right)\log\frac{(-4\pi^3)}{W''(\lambda_i)} - \frac{\hat{N}_i^2}{4}\log\frac{\hat{N}_i}{2\pi e^{3/2}} + \frac{\hat{N}_i}{4}\log\frac{2\hat{N}_i}{\pi^5 e}\right] - \\ &- \sum_{i<j}\hat{N}_i\hat{N}_j \log\left[2\pi(\lambda_i - \lambda_j)\right]. \end{aligned} \tag{5.42}$$

As explained above, we have $\mathcal{S}_i = \hat{N}_i$ since we are in the semiclassical approximation of the matrix model. Replacing $\hat{N}_i$ by $\mathcal{S}_i$ in (5.42), we can identify the quadratic terms in $\mathcal{S}_i$ with F_0^{np} and the linear terms with F_1^{np}:

$$F_0^{np} = -\sum_{i=1}^{d}\frac{\mathcal{S}_i^2}{4}\log\left[\frac{(-2\pi^2)\mathcal{S}_i}{e^{3/2}W''(\lambda_i)}\right] - \sum_{i<j}\mathcal{S}_i\mathcal{S}_j\log\left[2\pi(\lambda_i - \lambda_j)\right]\,, \tag{5.43}$$

$$F_1^{np} = -\sum_{i=1}^{d}\frac{\mathcal{S}_i}{4}\log\left[\frac{(-2\pi^8 e)}{\mathcal{S}_i W''(\lambda_i)}\right]. \tag{5.44}$$

According to (5.31), the non-perturbative contribution to the effective superpotential should be given by:

$$W_{\mathrm{eff}}^{np} = \sum_{i=1}^{d} N_i \frac{\partial F_0^{np}}{\partial \mathcal{S}_i} + 4F_1^{np} + \alpha\sum_{i=1}^{d}\mathcal{S}_i, \tag{5.45}$$

Ignoring constant and linear terms in $\mathcal{S}_i$ (which can be absorbed by a finite renormalization of the field theory scale Λ of Subsection 2.3), we find:

$$W_{\rm eff}^{np} = \sum_{i=1}^{d} \frac{\mathcal{S}_i}{2} \log \left[\frac{\Lambda^{2N-8} W''(\lambda_i)^{N_i+2} \prod_{j\neq i}(\lambda_i - \lambda_j)^{-2N_j}}{\mathcal{S}_i^{N_i-2}} \right] + O(\mathcal{S}) \,, \tag{5.46}$$

which recovers the leading contribution (2.26) derived by threshold matching.

6. Conclusions

We studied a class of non-anomalous, chiral $\mathcal{N}=1$ U(N) gauge theories with antisymmetric, conjugate symmetric, adjoint and fundamental matter in the context of the Dijkgraaf-Vafa correspondence. By using the method of generalized Konishi anomalies, we extracted a set of chiral ring constraints which allowed us to identify the 'dual' holomorphic matrix model and give an explicit expression for the gaugino superpotential in terms of matrix model data. This gives a proof of the Dijkgraaf-Vafa correspondence for a class of theories with quite nontrivial chiral matter content.

As a by-product of this analysis, we found that the effective superpotential of our models coincides with that produced upon confinement in non-chiral SO(N) field theories with a single chiral superfield transforming in the symmetric representation. This provides an independent proof of a relation suggested by holomorphy arguments.

Our results encourage us to think that similar methods could be applied successfully in order to extract non-perturbative information about more realistic models employed in supersymmetric phenomenology. It would be interesting to see how far this program can be implemented.

Surprisingly, we found that the number of fundamental flavors N_F in field theory must be taken to differ from the number of flavors $\hat{N}_F$ in the dual matrix model. This is unlike the non-chiral case considered in [11, 13, 14, 16, 15, 39, 8], for which the number of fundamental flavors agrees between field theory and the matrix model. More precisely, consistency of the matrix model requires $\hat{N}_F = 2$, while anomaly cancellation in our field theory requires $N_F = 8$. Despite this disagreement, one can match the Konishi constraints in the chiral ring with the loop equations of the two-flavor matrix model.

Another interesting question concerns the geometric engineering of our models, which provides their embedding in IIB string theory and leads to an interpretation of the gaugino superpotential as the flux-orientifold superpotential of certain non-compact Calabi-Yau backgrounds. As in the case of the U(N) theory with symmetric or antisymmetric matter [21, 22, 24] (whose geometric engineering was discussed in [25]), this can be achieved by applying the methods of [37]. A deeper understanding of the geometric realization could also shed light on the mismatch between the number of flavors in the matrix model and field theory. A detailed investigation of these issues is presented in [36].

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A. Gauge invariance of the matrix model measure

With an arbitrary number of flavors $\hat{N}_F$, we have the following transformations under the action (4.7):

$$\begin{aligned} d\hat{\Phi} &\to d\hat{\Phi} \\ d\hat{A} &\to (\det U)^{\hat{N}-1} d\hat{A} \\ d\hat{S} &\to (\det U)^{-(\hat{N}+1)} d\hat{S} \\ d\hat{Q} &\to (\det U)^{\hat{N}_F} d\hat{Q}\,, \end{aligned} \tag{A.1}$$

where U is an arbitrary element of $GL(\hat{N}, \mathbb{C})$. Therefore, the matrix model measure $d\mu = d\hat{\Phi} d\hat{A} d\hat{S} d\hat{Q}$ transforms as:

$$d\mu \to (\det U)^{\hat{N}_F - 2} d\mu\,, \tag{A.2}$$

and is invariant if and only if $\hat{N}_F = 2$.

The first relation in (A.1) is familiar from the (adjoint) holomorphic one-matrix model [4], while the last relation is obvious. Thus we only have to prove the second and third equation. For this, it suffices to check them for diagonalizable U, since our matter representation is continuous and the set of diagonalizable elements is dense in $GL(\hat{N}, \mathbb{C})$. Thus we can take $U = VTV^{-1}$, where V is an element of $GL(\hat{N}, \mathbb{C})$ and $T = \mathrm{diag}(t_1 \dots t_{\hat{N}})$, with $\prod_j t_j \neq 0$. The explicit form of the transformations (4.7) gives:

$$\begin{aligned} d\hat{A} &\to [\det\, R_a(U)]\; d\hat{A} \\ d\hat{S} &\to [\det\, R_{cs}(U)]\; d\hat{\Phi}\,, \end{aligned} \tag{A.3}$$

where R_a and R_{cs} are the antisymmetric and conjugate symmetric representations of $GL(\hat{N}, \mathbb{C})$:

$$\begin{aligned} R_a(U)(A) &= UAU^T\,, \qquad A^T = -A \\ R_{cs}(U)(S) &= U^{-T} S U^{-1} = U^{-T} S (U^{-T})^T\,, \qquad S^T = S\,. \end{aligned} \tag{A.4}$$

Using $R_a(VTV^{-1}) = R_a(V) R_a(T) R_a(V)^{-1}$ and $R_{cs}(VTV^{-1}) = R_{cs}(V) R_{cs}(T) R_{cs}(V)^{-1}$, we find:

$$\begin{aligned} d\hat{A} &\to [\det\, R_a(T)]\; d\hat{A} \\ d\hat{S} &\to [\det\, R_{cs}(T)]\; d\hat{S}\,. \end{aligned} \tag{A.5}$$

Since T is diagonal, one easily obtains:

$$\det R_a(T) = \prod_{1\leq i<j\leq \hat{N}} t_i t_j = \left(\prod_i t_i\right)^{\hat{N}-1}$$
$$\det R_{cs}(T) = \prod_{1\leq i\leq j\leq \hat{N}} t_i^{-1} t_j^{-1} = \left(\prod_i t_i\right)^{-(\hat{N}+1)} , \tag{A.6}$$

so that:

$$\det R_a(U) = (\det U)^{\hat{N}-1} \qquad \text{and} \qquad \det R_{cs}(U) = (\det U)^{-\hat{N}-1} . \tag{A.7}$$

This leads immediately to the second and third equation in (A.1).

B. Classical vacua of the SO(N) model

For the SO(N) model, the only matter is the complex superfield X, subject to the condition $X^T = X$ and transforming as follows under the gauge group:

$$X \rightarrow VXV^T = VXV^{-1} . \tag{B.1}$$

Here V is valued in SO($N, \mathbb{R}$). The F-flatness relations for $W_{\text{tree}} = \mathrm{tr}\, W(X)$ read:

$$W'(X) = 0 \tag{B.2}$$

while the D-flatness condition has the form:

$$\frac{1}{2i}[X, \bar{X}] = 0 , \tag{B.3}$$

where the left hand side is the moment map[12] for the action (B.1) of SO($N, \mathbb{R}$) (computed with respect to the natural symplectic form $\omega(X, Y) = -\mathrm{tr}(\bar{X}Y)$ on the representation space).

Writing $X = X^{re} + iX^{im}$, where X^{re} and X^{im} are two real-valued symmetric matrices, the D-flatness constraint becomes:

$$[X^{re}, X^{im}] = 0 , \tag{B.4}$$

which shows that X^{re} and X^{im} can be diagonalized simultaneously by performing a gauge transformation (B.1):

$$X^{re} = \mathrm{diag}(\lambda_1^{re}\mathbf{1}_{N_1}, \ldots, \lambda_D^{re}\mathbf{1}_{N_D})$$
$$X^{im} = \mathrm{diag}(\lambda_1^{im}\mathbf{1}_{N_1}, \ldots, \lambda_D^{im}\mathbf{1}_{N_D}) . \tag{B.5}$$

Here the real numbers λ_k^{re} and λ_k^{im} are such that the pairs $(\lambda_k^{re}, \lambda_k^{im})$ are mutually distinct.

[12] As usual, we used dualization with respect to the Killing form $(\xi, \eta) = -\mathrm{tr}(\xi\eta)$ of $o(N)$ in order to view the moment map as being valued in the Lie algebra $o(N)$ rather than in its dual.

Writing $\lambda_k := \lambda_k^{re} + i\lambda_k^{im}$, equations (B.6) show that X can be diagonalized via a gauge transformation (B.1):

$$X = \mathrm{diag}(\lambda_1 \mathbf{1}_{N_1}, \ldots, \lambda_D \mathbf{1}_{N_D})\,, \tag{B.6}$$

where λ_k are distinct complex numbers.

The F-flatness condition (B.2) now shows that we can take $D = d = \mathrm{deg} W'(z)$ and $\lambda_1 \ldots \lambda_d$ must coincide with the distinct critical points of W (again we use the convention that the block $\lambda_k 1_{N_k}$ does not appear in (B.6) if $N_k = 0$). In such a vacuum, the gauge group is broken down to $[\prod_{k=1}^{d} O(N_k)]/\{-1, 1\}$.

C. Derivation of the generalized Konishi constraints

Let us outline the derivation of the generalized Konishi relations. We will make heavy use of the chiral ring relations (3.5) and (3.7)–(3.10).

The first two equations (3.22)–(3.23) are straightforward modifications of the relations found in [7] for the U(N) model with a single adjoint field Φ and no extra matter. The only differences are due to the coupling of Φ to the symmetric and antisymmetric fields and appear on the left hand sides of these relations.

The derivation of the other relations is somewhat more involved. Let us consider first the symmetric field S and the general transformation:

$$\delta S = X^T S Y \tag{C.1}$$

where X and Y are arbitrary Grassmann even or odd $N \times N$ matrices. We start by computing the anomaly term induced by this transformation. For this, recall that $\mathcal{W}_\alpha$ acts on δS according to the conjugate symmetric representation:

$$\mathcal{W}_\alpha . \delta S = -\mathcal{W}_\alpha^T \delta S - \delta S \mathcal{W}_\alpha\,, \tag{C.2}$$

where juxtaposition on the right hand side stands for matrix multiplication.

Applying this formula twice, we find the anomaly term:

$$\begin{aligned} -32\pi^2 \mathcal{A} &:= \mathrm{Tr}\left(\mathcal{W}^\alpha . \mathcal{W}_\alpha . \frac{\partial \delta S}{\partial S}\right) \\ &= \frac{\partial(\delta S^{ir})}{\partial S^{ij}} \mathcal{W}^\alpha{}_r{}^s \mathcal{W}_{\alpha s}{}^j + 2\mathcal{W}^\alpha{}_r{}^i \frac{\partial(\delta S^{rs})}{\partial S^{ij}} \mathcal{W}_{\alpha,s}{}^j + \mathcal{W}^\alpha{}_r{}^i \mathcal{W}_{\alpha,m}{}^r \frac{\partial(\delta S^{mj})}{\partial S^{ij}} \\ &= \frac{1}{2}\mathrm{tr}(X)\,\mathrm{tr}(Y\mathcal{W}^2) + \frac{1}{2}\mathrm{tr}(XY\mathcal{W}^2) + \mathrm{tr}(\mathcal{W}^\alpha X)\,\mathrm{tr}(Y\mathcal{W}_\alpha) + \\ &\quad +(-)^{[X]}\mathrm{tr}(X\mathcal{W}^\alpha Y \mathcal{W}_\alpha) + \frac{1}{2}\mathrm{tr}(\mathcal{W}^2 X)\,\mathrm{tr}(Y) + \frac{1}{2}\mathrm{tr}(X\mathcal{W}^2 Y)\,. \end{aligned} \tag{C.3}$$

where $[X]$ denotes the Grassmann parity of X, $\mathrm{Tr}(\ldots)$ stands for the trace in the conjugate symmetric representation and $\mathrm{tr}(\ldots)$ is the trace in the fundamental representation.

Applying this relation for $X = Y = 1/z - \Phi$, we obtain:

$$\begin{aligned} \mathcal{A} &= -\frac{1}{32\pi^2}\left[\mathrm{tr}\,\frac{1}{z-\Phi}\,\mathrm{tr}\,\frac{\mathcal{W}^2}{z-\Phi} + 2\,\mathrm{tr}\,\frac{\mathcal{W}^2}{(z-\Phi)^2} + \mathrm{tr}\,\frac{\mathcal{W}^\alpha}{z-\Phi}\,\mathrm{tr}\,\frac{\mathcal{W}_\alpha}{z-\Phi}\right] \\ &= \mathbf{T}(z)\mathbf{R}(z) - 2\mathbf{R}'(z) - \frac{1}{2}\mathbf{w}^\alpha \mathbf{w}_\alpha\,. \end{aligned} \tag{C.4}$$

The left hand side of the generalized Konishi relation (3.12) involves the variation of the superpotential contracted with δS:

$$\begin{aligned}\delta S^{ij}\frac{\partial W}{\partial S^{ij}} &= -\frac{1}{2}\operatorname{tr}\left(\frac{1}{z-\Phi}A\frac{\Phi^T}{z-\Phi^T}S\right)+\frac{1}{2}\operatorname{tr}\left(\frac{\Phi}{z-\Phi}A\frac{1}{z-\Phi^T}S\right)+\\ &\quad +Q_f^T\frac{1}{z-\Phi^T}S\frac{1}{z-\Phi}Q_f\\ &= \operatorname{tr}\left(S\frac{1}{z-\Phi}A\right)+Q_f^T\frac{1}{z-\Phi^T}S\frac{1}{z-\Phi}Q_f\,.\end{aligned}$$

To arrive at this equation, we used the identity (4.30). Combining (C.5) and (C.4) leads to the Konishi relation (3.26).

We next consider the transformation:

$$\delta A = XAY^T\,. \tag{C.5}$$

Since:

$$\mathcal{W}_\alpha.\delta A = \mathcal{W}_\alpha\delta A + \delta A\mathcal{W}_\alpha^T\,, \tag{C.6}$$

this leads to the anomaly term:

$$\begin{aligned}-32\pi^2\mathcal{A} &:= \operatorname{Tr}\left(\mathcal{W}^\alpha.\mathcal{W}_\alpha.\frac{\partial\delta A}{\partial A}\right)\\ &= \mathcal{W}^\alpha{}_i{}^s\mathcal{W}_{\alpha s}{}^m\frac{\partial(\delta A_{mj})}{\partial A_{ij}}+2\mathcal{W}^\alpha{}_i{}^r\frac{\partial(\delta A_{rs})}{\partial A_{ij}}\mathcal{W}_{\alpha,j}{}^s+\frac{\partial(\delta A_{ir})}{\partial A_{ij}}\mathcal{W}^\alpha{}_s{}^r\mathcal{W}_{\alpha,j}{}^s\\ &= \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(Y\mathcal{W}^2)-\frac{1}{2}\operatorname{tr}(XY\mathcal{W}^2)+\operatorname{tr}(\mathcal{W}^\alpha X)\operatorname{tr}(Y\mathcal{W}_\alpha)-\\ &\quad -(-)^{[X]}\operatorname{tr}(X\mathcal{W}^\alpha Y\mathcal{W}_\alpha)+\frac{1}{2}\operatorname{tr}(\mathcal{W}^2X)\operatorname{tr}(Y)-\frac{1}{2}\operatorname{tr}(X\mathcal{W}^2Y)\,.\end{aligned} \tag{C.7}$$

Choosing $X=Y=\frac{1}{z-\Phi}$, one finds:

$$\begin{aligned}\mathcal{A} &= -\frac{1}{32\pi^2}\left[\operatorname{tr}\frac{1}{z-\Phi}\operatorname{tr}\frac{\mathcal{W}^2}{z-\Phi}-2\operatorname{tr}\frac{\mathcal{W}^2}{(z-\Phi)^2}+\operatorname{tr}\frac{\mathcal{W}^\alpha}{z-\Phi}\operatorname{tr}\frac{\mathcal{W}_\alpha}{z-\Phi}\right]\\ &= \mathbf{T}(z)\mathbf{R}(z)+2\mathbf{R}'(z)-\frac{1}{2}\mathbf{w}^\alpha\mathbf{w}_\alpha\,.\end{aligned} \tag{C.8}$$

On the other hand, the choice $X=\mathcal{W}^\beta/z-\Phi$ and $Y=\mathcal{W}_\beta/z-\Phi$ gives:

$$-32\pi^2\mathcal{A} = \operatorname{tr}\left(\frac{\mathcal{W}^\alpha\mathcal{W}^\beta}{z-\Phi}\right)\left(\frac{\mathcal{W}_\alpha\mathcal{W}_\beta}{z-\Phi}\right) = -(-32\pi^2)^2\frac{1}{2}\mathbf{R}(z)^2\,, \tag{C.9}$$

where we used the spinor identities:

$$\begin{aligned}\psi^\alpha\psi^\beta &= \frac{1}{2}\epsilon^{\alpha\beta}\psi^2\,,\\ \psi_\alpha\psi_\beta &= -\frac{1}{2}\epsilon_{\alpha\beta}\psi^2\,,\\ \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} &= \delta_\alpha^\gamma\,.\end{aligned} \tag{C.10}$$

The left hand side of the anomaly equation for $X = Y = \frac{1}{z-\Phi}$ has the form:

$$\delta A_{ij}\frac{\partial W}{\partial A_{ij}} = \mathrm{tr}\left(S\frac{1}{z-\Phi}A\right), \tag{C.11}$$

where again we used identity (4.30). Together with (C.8). this leads to the Konishi relation (3.24).

If we chose $X = \mathcal{W}^\beta/z-\Phi$ and $Y = \mathcal{W}_\beta/z-\Phi$, the left hand side becomes:

$$\delta A_{ij}\frac{\partial W}{\partial A_{ij}} = \frac{1}{2}\,\mathrm{tr}\left(S\frac{\Phi\mathcal{W}^\beta}{z-\Phi}A\frac{\mathcal{W}_\beta^T}{z-\Phi^T}\right) - \frac{1}{2}\,\mathrm{tr}\left(S\frac{\mathcal{W}^\beta}{z-\Phi}A\frac{\Phi^T\mathcal{W}_\beta^T}{z-\Phi^T}\right). \tag{C.12}$$

We next notice that:

$$\mathrm{tr}\left(S\frac{\mathcal{W}^\beta}{z-\Phi}A\frac{\mathcal{W}_\beta^T}{z-\Phi^T}\right) = -\,\mathrm{tr}\left(\frac{\mathcal{W}_\beta}{z-\Phi}A^T\frac{(\mathcal{W}^\beta)^T}{z-\Phi^T}S^T\right) = -\,\mathrm{tr}\left(S\frac{\mathcal{W}^\beta}{z-\Phi}A\frac{\mathcal{W}_\beta^T}{z-\Phi^T}\right) = 0 \tag{C.13}$$

where in the first line we used invariance of the trace under transposition and in the second line we used the symmetry of S and antisymmetry of A together with the spinor identity $\psi^\beta\psi_\beta = -\psi_\beta\psi^\beta$. A similar derivation shows that:

$$\begin{aligned} -\,\mathrm{tr}\left(S\frac{\mathcal{W}^\beta}{z-\Phi}A\frac{\Phi^T\mathcal{W}_\beta^T}{z-\Phi^T}\right) &= \mathrm{tr}\left(S\frac{\Phi\mathcal{W}^\beta}{z-\Phi}A\frac{\mathcal{W}_\beta^T}{z-\Phi^T}\right) \\ \mathrm{tr}\left(S\frac{\mathcal{W}^\beta}{z-\Phi}A\frac{(z-\Phi^T)\mathcal{W}_\beta^T}{z-\Phi^T}\right) &\equiv -\,\mathrm{tr}\left(S\frac{\mathcal{W}^2}{z-\Phi}A\right) \end{aligned} \tag{C.14}$$

where in the last line we used the chiral ring relation $A\mathcal{W}_\beta^T \equiv -\mathcal{W}_\beta A$. Combining this with the anomaly term (C.9) leads to the Konishi relation (3.25).

Let us now turn to transformations of the fundamental fields. Starting with:

$$\delta Q_f = \sum_{g=1}^{\hat{N}_F}\lambda_{fg}\frac{1}{z-\Phi}Q_g\,, \tag{C.15}$$

we find:

$$2\sum_{g=1}^{\hat{N}_F}\lambda_{fg}Q_f^T S\frac{1}{z-\Phi}Q_g \equiv \mathbf{R}(z)\lambda_{ff} \tag{C.16}$$

where we do *not* sum over the flavor index f. This is the most general form of the Konishi anomaly without any restriction on λ_{fg}. Since λ_{fg} is arbitrary, we conclude:

$$2Q_f^T S\frac{1}{z-\Phi}Q_g \equiv \mathbf{R}(z)\delta_{fg}\,. \tag{C.17}$$

Taking the trace of this equation leads to the Konishi relation (3.28).

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